Growth of cocycles of Isom. rep. on Banach spaces
Séance 2

A-T-menability   proper = metrically proper

$b : G \to E$

$\sigma : G \curvearrowright E$ F bounded

$\{g \in G \mid gF \cap F \neq \emptyset\}$ is rel. compact

$$\lim_{g \to \infty} \|b(g)\| = +\infty$$

LEMMA:

E is (p, C_E)-unif. convex metric space
$A \subset B \subset E$ bounded sets

$$C_E \, d(c(A), c(B))^p \leqslant \rho(B)^p - \rho(A)^p$$

LEMMA (Lafforgue)

Given E a (p, C_E)-unif. convex metric space and an isometric continuous action $G \curvearrowright E$ w/o fixed pts where G is a loc. compact second countable with S a rel. compact generating set ($S = \bar{S}^{-1}$)

For every $N \in \mathbb{N}$, $\varepsilon > 0$, there exists $v \in E$ such that:

$$\rho(S^n . v) \geqslant \left(C_E (1-\varepsilon)(n-1) + 1\right)^{1/p} \rho(S^1 . v)$$

for every $n \in \{1, 2, \dots, N\}$

Pf:

For any $v \in E$, let's denote $c_n(v) = c(S^n . v)$
$\rho_n(v) = \rho(S^n . v)$

the idea of the proof will be to suppose that the lemma is false and prove:

Claim: For any $v \in E$, there exists $v' \in E$ s.t
$\rho_1(v') \leqslant (1-\varepsilon)^{1/p} \rho_1(v)$ and $d(v, v') \leqslant M \rho_1(v)$

for some $M > 0$ (indep. of v)

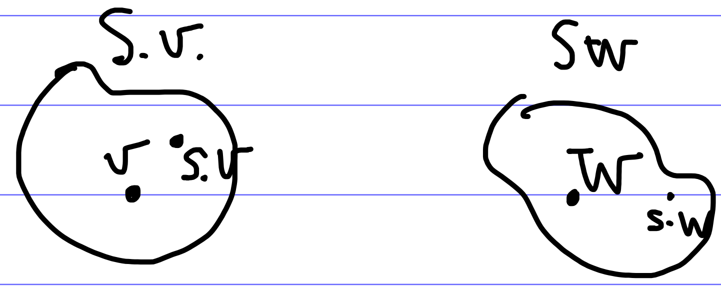
<u>Pf</u> of lemma assuming claim:

The claim will imply that there exists $(v_n)_{n \geq 0} \in E$ s.t

$$\rho_1(v_n) \leqslant (1-\varepsilon)^{n/p} \rho_1(v_0)$$
$$d(v_n, v_{n+1}) \leqslant M \rho_1(v_n)$$

(v_n) is a Cauchy sequence
Let $\bar{v}$ be its limit. Let's note that ρ_1 is 1-Lipschitz, since $S.v \subset [S.w]_{d(v,w)} \subset B(c_1(w), \rho_1(w) + d(v,w))$

$$[X]_r = \bigcup_{x \in X} B(x, r)$$



$$d(s.v, s.w) = d(v,w)$$
$$\Rightarrow S.v \subset [S.w]_{d(v,w)}$$

$$S.v \subset B(c_1(w), \rho_1(w) + d(v,w))$$
$$\rho_1(v) \leqslant \rho_1(w) + d(v,w)$$
$$\rho_1(v) - \rho_1(w) \leqslant d(v,w)$$
$$|\rho_1(v) - \rho_1(w)| \leqslant d(v,w)$$

$$\rho_1(v_n) \to 0 \Rightarrow \rho_1(\bar{v}) = 0$$

$$S.\bar{v} = \bar{v} \Rightarrow \bar{v} \text{ is a fixed point}$$

<u>Pf</u> (Claim)
Supposing the lemma is false, we have that there exists $N \in \mathbb{N}$, $\varepsilon > 0$ s.t. for every $v \in E$, $\exists m \in \{2, \dots, N\}$ such that

$$\rho_m(v)^p < (C_E(1-\varepsilon)(m-1)+1) \; \rho_1(v)^p$$

There exists $n \in \{1, 2, \dots, m-1\}$ s.t:

$$\rho_{n+1}(v)^p - \rho_n(v)^p \leqslant C_E(1-\varepsilon)\rho_1(v)^p \quad (A)$$

We know $\underbrace{g.S^n}_{A}.v \subset \underbrace{S^{n+1}}_{B}.v$ for all g in S

$$C_E\, d(c(A), c(B))^p \leqslant \rho(B)^p - \rho(A)^p$$

$$C_E\, d(g.c_n(v), c_{n+1}(v))^p \leqslant \rho_{n+1}(v)^p - \rho_n(v)^p$$

$$\leqslant C_E(1-\varepsilon)\ \rho_1(v)^p$$

$$d(g.c_n(v), c_{n+1}(v)) \leqslant (1-\varepsilon)^{1/p}\ \rho_1(v)$$

for all $g \in S$

$$S.c_n(v) \subset B(c_{n+1}(v), (1-\varepsilon)^{1/p}\rho_1(v))$$

$$v' = c_n(v), \quad \rho_1(v') \leqslant (1-\varepsilon)^{1/p}\ \rho_1(v)$$

$$d(c_n(v), v) \leqslant \rho_n(v) \leqslant \rho_m(v) \leqslant (C_E(1-\varepsilon)(m-1)+1)^{1/p}\rho_1(v)$$

$$\leqslant (C_E N+1)^{1/p}\rho_1(v)$$

$$M = (C_E N+1)^{1/p}$$

cond. negative functions $= \{\varphi : G \to \mathbb{R}$, given by $\varphi(g) = \|b(g)\|^2$ for some cocycle $b \in Z^1(G,\pi)\}$

GNS construction.

$\varphi_i \to \varphi$ unif. on compacts, φ_i cond. neg. functions $\Rightarrow \varphi$ cond. neg. function

Furthermore

$$a_n(v) = \sup_{g \in S^n} d(g.v, v)$$

$$= \sup_{|g| \leqslant n} d(g.v, v)$$

We have :

$$2\rho_n(v) \geqslant a_{2n}(v) \geqslant a_n(v) \geqslant \rho_n(v)$$

$S^n . v \subset B(c_n(v), \rho_n(v))$

$\mathrm{diam}\,(S^n . v) = a_{2n}(v)$ $\qquad a_{2n}(v) \leqslant 2\rho_n(v)$

$S^n . v \subset B(v, a_n(v))$

$\rho_n(v) \leqslant a_n(v)$

Cor: Under same conditions of Lafforgue

$$a_n(v) \geqslant (C_E(1-\varepsilon)(n-1)+1)^{1/p} \frac{a_2(v)}{2}$$

since $\rho_1(v) \geqslant \frac{a_2(v)}{2}$ $\quad \left(\frac{\sqrt{n}}{2}\right) \to$ Lafforgue

2) Ultraproducts of Representations

$\mathcal{U}$ non-principal ultrafilter on $\mathbb{N}$

$$\lim_{n\to\mathcal{U}} : \ell^\infty(\mathbb{N}) \longrightarrow \mathbb{R}$$

$$a_n \longmapsto \lim_{n\to\mathcal{U}} (a_n)$$

$$\lim_{n\to\mathcal{U}} (a_n b_n) = \lim_{\mathcal{U}} a_n \cdot \lim_{\mathcal{U}} b_n$$

$$\lim_{n\to\mathcal{U}} (a_n + b_n) = \lim_{n\to\mathcal{U}} a_n + \lim_{n\to\mathcal{U}} b_n$$

E, $\quad \prod_{\mathcal{U}} E = \ell^\infty(\mathbb{N}, E)/\sim$

$(a_n) \sim (b_n) \quad \lim_{n\to\mathcal{U}} \| a_n - b_n \| = 0$

$\ell^\infty(\mathbb{N}, E) = \{ (a_n)_{n\in\mathbb{N}} \mid \sup_{n\in\mathbb{N}} \| a_n \| < \infty \}$

Heinrich $\quad E \ L^p(\Omega, \mu) \Rightarrow \prod_{\mathcal{U}} E \quad L^p(\Omega', \mu')$

$\pi_i : G \to \mathcal{O}(E)$ isometric rep.
continuous
strongly cont. SOT on $\mathcal{O}(E)$

$\pi : G \longrightarrow \mathcal{O}(\prod_{\mathcal{U}} E)$ won't be strongly continuous

$[0, +\infty[\to B(E)$ strongly continuous
semigroups
Cherix,

M. de la Salle. A local characterization of Kazhdan's projections and applications 2019

$E_{\mathcal{U}}$ = closure of $\{\pi(f)x, \ x \in \prod_{\mathcal{U}} E_i \ f \in C_c(G)\}$

$\pi : G \to \mathcal{O}(\prod_{\mathcal{U}} E)$

$\pi_{\mathcal{U}} : G \to \mathcal{O}(E_{\mathcal{U}})$ strongly continuous

Prop 4.11 [de la salle]

$E_{\mathcal{U}}$ is an L^p-space if E L^p-space

<u>Prop</u> (de la Salle):

$\sigma_i : G \curvearrowright E$ cont. isom. actions on E

$\sigma_i(g)v = b_i(g) + \pi_i(g).v$ $\quad \pi_i : G \to \mathcal{O}(E)$

Assume that:

i) b_i are pointwise bounded

$$\sup_{i \in \mathbb{N}} \|b_i(g)\| < \infty$$

ii) b_i equicontinuous at identity of G

$\forall\, \varepsilon>0$, $\exists\, U\subset G$ neighbourhood of the id. s.t.

$$\sup_{i\in\mathbb{N}} \sup_{g\in U} \|b_i(g)\| < \varepsilon$$

We then have that there exists $G \curvearrowright E_{\mathcal{U}}$ continuous affine isometric action with linear part $\pi_{\mathcal{U}}$ and translation part $b(g)=(b_i(g))_{\mathcal{U}}$

And we have that:

$$\sup_{|g|\leq n} \|b_{\mathcal{U}}(g)\| = \lim_{\mathcal{U}} \sup_{\|g\|\leq n} \|b_i(g)\|$$

<u>Thm</u>(P, A. Lopez Neumann 2024)

Given G without FL^p, $\exists\, E$ L^p-space together with $\pi: G\to \mathcal{O}(E)$ strongly continuous and $b: G\to E$, $b\in Z^1(G,\pi)$ s.t.

$$\sup_{|g|\leq n} \|b(g)\| \geq \left(\frac{1}{2}\left(C_p(n-1)+1\right)^{1/p}-2\right)\sup_{|g|\leq 1}\|b(g)\|$$

for all $n\geq 1$, $\sup_{|g|\leq 1}\|b(g)\|>0$

<u>Pf</u>:

$\sigma: G\curvearrowright E$ isom. action, $v\in E$

$b(g)=\sigma(g).v-v \in Z^1(G,\pi)$

$b(gh)=b(g)+\pi(g)b(h)$

$G\curvearrowright F$ w/o fixed points, F L^p-space

linear part $\pi: G\to \mathcal{O}(F)$

$b: G\to F$

$\exists\, (v_i)_{i\in\mathbb{N}}$ s.t.

$$\sup_{|g|\leq n} \|g.v_i-v_i\| \geq \frac{1}{2}\left(C_p\left(1-\frac{1}{i}\right)(n-1)+1\right)^{1/p} \sup_{|g|\leq 2}\|g.v_i-v_i\|$$

for all $n \in \{1, \dots, i\}$, (Corollary $\varepsilon = 1/i$, $N = i$)

We can renormalize b and v_i in order to get new actions $\sigma_i : G \curvearrowright F$ and vectors $\overline{v_i} \in F$ $\overline{b_i}(g) = \sigma_i(g).\overline{v_i} - \overline{v_i}$ in order to obtain

$$\sup_{|g| \leq 2} \|\overline{b_i}(g)\| = 1$$

For all $n \leq i$:

$$\sup_{|g| \leq n} \|\overline{b_i}(g)\| \geqslant \frac{1}{2}\left(C_p\left(1 - \frac{1}{i}\right)(n-1) + 1\right)^{1/p} \sup_{|g| \leq 2} \|\overline{b_i}(g)\|$$

Consider m_G a left-inv. Haar measure

$\chi : G \to [0, +\infty[$ of compact support continuous

$\operatorname{supp} \chi \subset V$, neighbourhood of id. $V.V \subset S$

$V.S.V \subset S^2$

$V = V^{-1}$

$$\int_G \chi(g) \, dm_G(g) = 1$$

$$\overline{b_i}(g) = \sigma_i(g).\overline{v_i} - \overline{v_i}$$

$$\widetilde{v_i} = \int_G \chi(h) \, \sigma_i(h).\overline{v_i} \, dm_G(h)$$

$$b_i(g) = \sigma_i(g).\widetilde{v_i} - \widetilde{v_i}$$

Let's first prove that:

$$\sup_{|g| \leq 1} \|b_i(g)\| \leqslant 1$$

This is true since for all g in S:

$$\| b_i(g) \| = \| \sigma_i(g).\tilde{v}_i - \tilde{v}_i \|$$

$$= \left\| \int_V \int_V \chi(h_1)\chi(h_2) [\sigma_i(gh_1)\bar{v}_i - \sigma_i(h_2)\bar{v}_i] \, dh_1 \, dh_2 \right\|$$

$$\leqslant \int_V \int_V \chi(h_1)\chi(h_2) \| \sigma_i(h_2^{-1} g h_1).\bar{v}_i - \bar{v}_i \| \, dh_1 \, dh_2$$

$VSV \subset S^2$, $\quad \sup_{|g|_S \le 2} \| \sigma_i(g).\bar{v}_i - \bar{v}_i \| = 1$

$\leqslant 1.$

$$\| b_i(g) \| \leqslant |g|_S \sup_{|g| \leqslant 1} \| b_i(g) \| \leqslant |g|_S$$

First condition (i) ✓

Let's consider $\varepsilon > 0$

$$\| b_i(g) \| = \left\| \int_G \chi(h)\sigma_i(gh).\bar{v}_i \, dh - \int_G \chi(h)\sigma_i(h).\bar{v}_i \right\|$$

$$= \left\| \int_G \chi(g^{-1}h)\sigma_i(h).\bar{v}_i \, dh - \int_G \chi(h)\sigma_i(h).\bar{v}_i \, dh \right\|$$

$$= \Big\| \int_G \chi(g^{-1}h) [\sigma_i(h).\bar{v}_i - \sigma_i(g).\bar{v}_i] \, dh +$$

$$\int_G \chi(h) [\sigma_i(g).\bar{v}_i - \sigma_i(h).\bar{v}_i] \, dh \Big\|$$

$$= \left\| \int_G [\chi(g^{-1}h) - \chi(h)] [\sigma_i(h).\bar{v}_i - \sigma_i(g).\bar{v}_i] \, dh \right\|$$

$$\leqslant \int_G |\chi(g^{-1}h) - \chi(h)| \, \| \sigma_i(h^{-1}g).\bar{v}_i - \bar{v}_i \| \, dh$$

$$\leqslant \int_G |\chi(g^{-1}h) - \chi(h)| \, |h^{-1}g|_S \, dh$$

U nbd. of id s.t. $\quad \int_G |\chi(g^{-1}h) - \chi(h)| < \varepsilon \quad \forall g \in U$

$\underline{U \subset S}$

$g \in U \qquad h \notin gV \cup V \Rightarrow \quad x(g^{-1}h) - x(h) = 0$

$h \in gV \cup V, \qquad |h^{-1}g|_S \leq 2$

$\forall g \in U \Rightarrow \| b_i(g) \| \leq 2\varepsilon$

$$\|\tilde{v}_i - \overline{v}_i\| = \left\| \int_V x(h) \left[\sigma_i(h).\overline{v}_i - \overline{v}_i\right] dh \right\|$$

$$\leq \int_V x(h)\, dh = 1$$

$$\sup_{|g| \leq n} \|b(g)\| \geq \lim_{i \to \mathcal{U}} \left(\frac{1}{2} \left(c_p \left(1 - \frac{1}{i}\right)(n-1) + 1 \right)^{1/p} - 2 \right)$$

$$= \left(\frac{1}{2} \left(c_p (n-1) + 1 \right)^{1/p} - 2 \right) . 1$$

$$\geq \left(\frac{1}{2} \left(c_p (n-1) + 1 \right)^{1/p} - 2 \right) \sup_{|g| \leq 1} \|b(g)\|$$

$\blacksquare$