Growth of cocycles for Isom. Rep. on Banach spaces

Let's fix some notations around isom. rep. on real Banach spaces:

Given E a real Banach space

$$\mathcal{O}(E) = \{T : E \to E \mid T \text{ isometric linear surjective}\}$$

$$\mathrm{Isom}(E) = \{f : E \to E \mid f \text{ isometry of } E\}$$

By Mazur_Ulam's theorem, it's known that

$$\mathrm{Isom}(E) = E \rtimes \mathcal{O}(E)$$

that is for any $f : E \to E$ isometry is of the form:

$$f(v) = b + Av \quad \text{for some } A \in \mathcal{O}(E),\ b \in E$$

We say that E is an L^p-space if $E = L^p(\Omega, \mu)$ for (Ω, μ) std. measured space

Moreover, we have Banach-Lamperti's thm:

For $1 < p \neq 2$, any $U \in \mathcal{O}(L^p(\Omega, \mu))$, there exists $T : (\Omega, \mu) \to (\Omega, \mu)$ non-singular such that

$$Uf(x) = f(T(x)) \cdot h(x) \left(\frac{d\, T_* \mu}{d\mu}(x) \right)^{1/p}$$

$$h : X \to \mathbb{R} \quad |h(x)| = 1 \quad \mu\text{-a.e.}$$

for every $f \in L^p(\Omega, \mu)$.

In the case of a Hilbert space $\mathcal{H}$, $\mathcal{O}(\mathcal{H}) = U(\mathcal{H})$

G loc. compact second countable gp
(lcsc)
compactly generated, $\exists\, S$ compact

$$G = \bigcup_{n=1}^{\infty} S^n, \quad S = S^{-1}$$

$$|g|_S = \min\{k \mid g = s_1 \dots s_k,\ s_i \in S\}$$

DEF: (Isom. rep.)

We say that a morphism $\pi: G \to \mathcal{O}(E)$ is an isom. rep if it's continuous w.r.t. SOT
strong op. top

E $\quad B(E)$ $\quad$ norm topology $\|.\|$
SOT $\quad B(E) \to E$ cont. $\quad T \mapsto Tv$
WOT

$B(E) \to E$ cont
$T \mapsto \langle Tv, w\rangle$
$w \in E'$
$v \in E$

Now given a isometric. rep $\pi: G \to \mathcal{O}(E)$ we say that a measurable $b: G \to E$ is 1-cocycle (w.r.t π) if

$$b(gh) = b(g) + \pi(g) b(h)$$

for all $g, h \in G$

PROP: If $b(G) \subset E$ is separable, then $b: G \to E$ is continuous

· Moreover, these cocycles induce · isom. actions of G on E.

$$\sigma_g(v) = b(g) + \pi(g).v$$

b 1-cocycle for $\pi \Leftrightarrow \sigma_{gh} = \sigma_g \sigma_h$

· Let's denote by $Z^1(G,\pi)$ the set of all 1-cocycles of π.

Moreover σ has a fixed pt. $\Rightarrow$ b is bounded

($\Rightarrow$) $\sigma_g(v) = v \quad \forall g$

$$b(g) + \pi(g).v = v$$

$$b(g) = v - \pi(g).v$$

$$\|b(g)\| \leq 2\|v\|$$

"($\Leftarrow$)" b bounded

$\sigma_g(G).v$ bounded

Circumcenter arguments

$\exists$ fixed pt. $\begin{cases} \text{unif. convex Banach spaces} \\ \text{Hilbert} \\ L^p(\Omega,\mu) \quad 1<p<\infty \end{cases}$

<u>Def:</u> We say that G has prop. (FH) if any cont. isom. action on any Hilbert space has a fixed pt, this is the same as saying that any 1-cocycle for any π is bounded.

For σ-compact groups, Delorne - Guichardet thm $\Rightarrow$ (FH) = (T)

L^p-analogues

<u>Def</u>: G has FL^p if any cont. isom. affine act$^\circ$ on any L^p-space has a fixed pt.

$FL^2 = FH$.

$FL^p \sim FH \qquad 1 < p < 2$

· M. de la Salle, A. Marrakchi (2021) $\Rightarrow$ $FL^p \underset{p>q}{\Rightarrow} FL^q$

$G \to G/N \quad N \lhd G$

G has $FL^p \Rightarrow G/N$ has FL^p

<u>Def</u>: G is A-FL^p-menable if $\exists$ proper isom. act$^\circ$ on L^p-space

A-FL^2-menable = Haagerup a-T-menable

A-FL^p-menable $\supset$ amenable gps

A-FL^p-menable + FL^p = G compact

(*) $a: G \times E \to E \qquad K$ bounded
$G_K = \{g \in G \mid gK \cap K \neq \emptyset\}$ is precompact

· A-FL^p-menable $\underset{p<q}{\Rightarrow}$ A-FL^q-menable

$H < G \qquad G$ A-FL^p-menable $\Rightarrow H$ A-FL^p-menable

Examples:

G simple k Lie-group $\quad k$ local field
$= \mathbb{R}, \mathbb{Q}_p, \mathbb{F}_q((t))$

- G $\mathrm{rk}\, G \geq 2$ $\quad$ G has FL^p $\forall p \geq 2$

- G $\mathrm{rk}_{\mathbb{R}} G = 1$

$\left.\begin{array}{l} SO(n,1) \\ SU(n,1) \end{array}\right\}$ Haagerup

$\left.\begin{array}{l} Sp(n,1) \\ F_4^{-20} \end{array}\right\}$ prop. (T) $\quad$ A-FL^p-menable $p > 4n+2$
A-FL^p-menable $p > 22$

These notions are not inv. by Q.I!

prop (T) $\quad$ Bekka, dlH, Valette
Haagerup $\quad$ Carette

Growth of cocycles:

<u>Thm</u> (V. Lafforgue)

If G doesn't have (FH), then $\exists \pi, b \in Z^1(G, \pi)$ such that

$$\sup_{|g|_S \leq n} \|b(g)\| \geq \left(\frac{\sqrt{n}}{2} - 2\right) \sup_{|g|_S \leq 1} \|b(g)\|$$

and $\sup_{|g|_S \leq 1} \|b(g)\| > 0$

G discrete $\quad \left(\frac{\sqrt{n}}{2} - 2\right)$ by $\frac{\sqrt{n}}{2}$

$b \in Z^1(G,\pi)$

$$b(gh) = b(g) + \pi(g)\, b(h)$$

$$\|b(gh)\| \leqslant \|b(g)\| + \|b(h)\|$$

$$\|b(g)\| \leqslant |g|_S \sup_{|g|\leq 1} \|b(g)\|$$

V. Lafforgue: $G = SO(n,1)$, $SL(2,\mathbb{Q}_p)$

$\forall_\pi\, b \in Z^1(G,\pi)$ $\lim_{g\to\infty} \dfrac{\|b(g)\|^2}{|g|+1}$ exists

<u>Thm</u> (P., A. Lopez-Neumann 2024)

If G doesn't have FL^p, then $\exists\, b, \pi$

$b \in Z^1(G,\pi)$, $\pi: G \to O(E)$ E L^p-space

such that

$$\sup_{|g|\leq n} \|b(g)\| \geqslant \left(\frac{1}{2}\left(C_p(n-1)+1\right)^{1/p} - 2\right) \sup_{|g|\leq 1} \|b(g)\|$$

C_p constant depending only on p

Unif. Convex Metric Spaces:

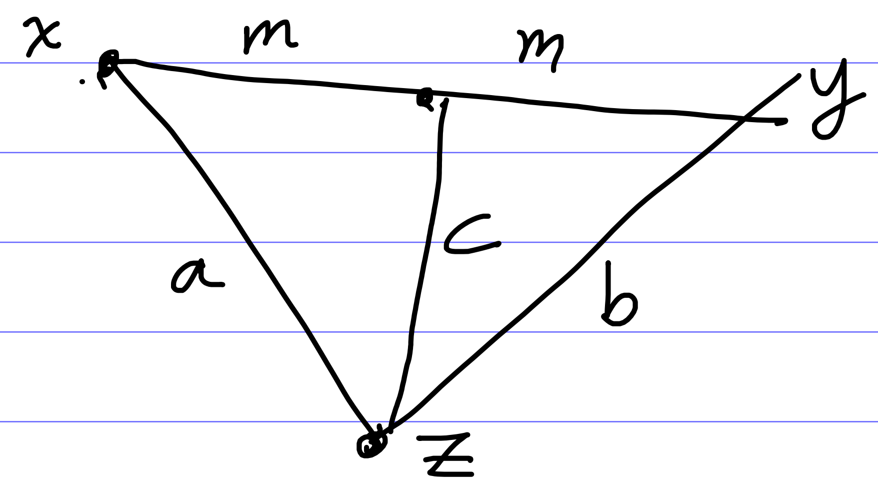
We say that (E,d) is uniquely geodesic if $\forall x,y \in E$, $\exists!\, \gamma$ geodesic joining x to y

$\gamma: [0,1] \to E$ ($d(\gamma(s),\gamma(t)) = |s-t|\, d(x,y)$ $\forall s,t$)

<u>DEF</u>: (E,d) is a unif. convex metric space if it is complete, uniquely geodesic and for all $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $\forall\, x,y,z \in E$ with $d(x,y) \geqslant \varepsilon \max(d(x,z), d(y,z))$ we have

$$d(z, m_{xy}) \leqslant (1-\delta) \max(d(x,z), d(y,z))$$

m_{xy} is the middle point of x and y



$E = \mathbb{R}^2$

$$c^2 + m^2 = \frac{1}{2}(a^2+b^2)$$

$c \leq \max\{a, b\}$

DEF: (E, d) is a (p, C_E)-unif. convex metric space if $\forall x, y, z$

$$(A) \quad d(z, c_t)^p \leq t\, d(z,x)^p + (1-t) d(z,y)^p - C_E\, t(1-t)\, d(x,y)^p$$

c_t geodesic from x to y

$c_0 = x, \quad c_1 = y$

Examples: $1 < p < \infty$

- $L^p(\Omega, \mu)$ $\begin{cases} \text{is } p\text{-unif. convex} & p \geq 2 \\ 2\text{-unif. convex} & 1 < p < 2 \end{cases}$

- (M, g) $\pi_1(M) = 0$, $\sec \leq 0$ 2-unif. convex
- CAT(0)-spaces
- Finsler-geometry (?)
- ℓ^p-metrics in cell complexes $\to$ p-unif (Hoda, Pettyt, Haettel 2023)

E p-unif. convex $\Rightarrow$ reflexive Banach spaces

superreflexive $\Rightarrow$ $\exists\ \|.\|$ p-unif. convex

Pisier: E Banach space unif. convex $\Rightarrow$ p-unif. convex for some p

In (p, C_E)-unif. convex spaces we can define the notion of "circum center" and "circum radius"

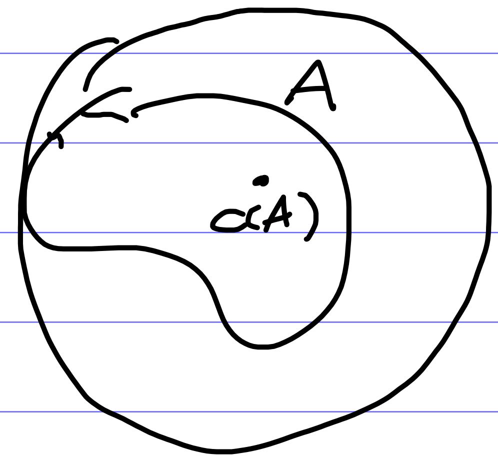
If $A \subset E$ Bounded set

$$\rho(A) = \inf \{ r \mid \bar{B}(p, r) \supset A \}$$

$$c(A) = c \text{ s.t } A \subset \bar{B}(c, \rho(A))$$

$\bar{B}(c(A), \rho(A))$ is ball of minimal radius containing A.

inf is a min (completeness)
uniqueness of $c(A)$ comes from (A)



$$x \in E, \quad r(x) \text{ s.t. } B(x, r(x)) \supset A$$

$$\rho(A) = \inf_{x \in E} r(x)$$

<u>Prop</u>: If (E, d) is (p, C_E)-unif convex

$A \subset B \subset E$ bounded, we have that

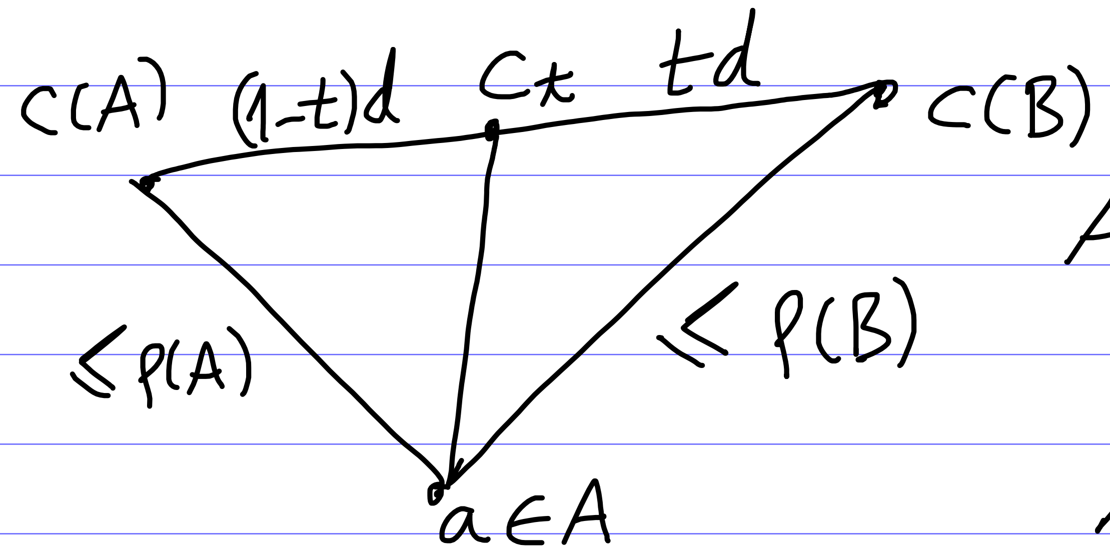
$$c_E \, d(c(A), c(B))^p \leq \rho(B)^p - \rho(A)^p$$

<u>Pf</u>: $A \subset B \subset B(c(B), \rho(B))$

$$A \subset B(c(B), \rho(B)) \Rightarrow \rho(A) \leq \rho(B)$$

If $c(A) = c(B)$, nothing to prove
Let's suppose $c(A) \neq c(B)$.

$d = d(c(A), c(B))$



$A \subset B(c(A), \rho(A))$
$d(a, c(A)) \leq \rho(A)$

$A \subset B(c(B), \rho(B))$
$d(a, c(B)) \leq \rho(B)$

$$d(a, c_t)^p \leq (1-t)\, d(a, c(A))^p + t\, d(a, c(B))^p - c_E d^p t(1-t)$$

$$\leq \underbrace{(1-t)\,\rho(A)^p + t\,\rho(B)^p - c_E t(1-t)\, d^p}_{h(t)}$$

$$h(t) = (c_E d^p) t^2 - (\rho(A)^p - \rho(B)^p + c_E d^p)\, t + \rho(A)^p$$

Suppose $\quad c_E d^p > \rho(B)^p - \rho(A)^p$

$h(t)$ has it's minimal value at

$$t_0 = \frac{\rho(A)^p - \rho(B)^p + c_E d^p}{2 c_E d^p}$$

$t_0 \in [0, 1/2]$
$h(t_0) < \rho(A)^p$

We then have that for all a in A

$$d(a, c_{t_0})^p \leq h(t_0) < \rho(A)^p$$

$$A \subset B(c_{t_0}, (h(t_0))^{1/p})$$

Contradiction since $\quad h(t_0)^{1/p} < \rho(A)$
$\rho(A)$ minimal radius.