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MASTER MATHÉMATIQUE ET APPLICATIONS
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Mémoire de Master 2

DEHN FUNCTIONS OF NILPOTENT GROUPS

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Abstract

In this mémoire we will review some results of the article of R.Young [10], mainly a theorem which under some good conditions allows us to prove bounds on the Filling Volume functions and Higher Dehn functions of some latices in Carnot groups. As a consequence of that R.Young gives quadratic bounds for the Dehn function of a certain family of 2-step nilpotent groups, and gives euclidean bounds for higher Dehn functions of Jet groups; a family of groups based on the jet space of $\mathbb{R}^k$, which generalizes the Heisenberg groups. Moreover, the original part of this mémoire consists in finding a family of 3-step nilpotent groups for which these methods also work giving us cubic bounds for these groups(Section 3.3).

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Chapter 1

Preliminaries

1.1 Dehn Function

The objective of this mémoire is to study the Dehn functions and filling volume functions of nilpotent groups, in order to do that we first need to define what are these. Moreover, there are several possible definitions for these functions, but they are all equivalent in a coarse sense. To make this rigorous, we can define a partial ordering on functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ so that $f \preceq g$ if and only if there exists c such that:

$$f(x) \leq cg(cx+c) + cx + c$$

And we say that f and g are equivalent in a coarse sense, which we will denote $f \sim g$, if $f \preceq g$ and $g \preceq f$.

Definition 1.1. (*Dehn function*)

Let G be a finitely presented group, with presentation $\langle S|R\rangle$, we say that a nullhomotopic word $w \in F(S)$ has area $A(w) = N$, if N is the smallest integer such that:

$$w = w_1 r_1 w_1^{-1} w_2 r_2 w_2^{-1} \dots w_N r_N w_N^{-1}$$

in $F(S)$; where w_i is in $F(S)$, and r_i is in R. The **Dehn function** of G is:

$$\delta_G(n) = \sup_{|w|_S \leq n} A(w)$$

The Dehn function can also be given a more geometric definition, in terms of riemannian filling function, as follows.

Definition 1.2. (*Filling function*) Let G be a finitely presented group, acting proper cocompactly by isometries on a simply connected riemannian manifold X, for a given Lipschitz loop γ in X, we define the area $A(\gamma)$ of γ as the infimum of the areas of Lipschitz discs in X with boundary γ. We then define the **Filling function** of M as:

$$Fill_M(t) = \sup_{l(\gamma) \leq t} A(\gamma)$$

where $l(\gamma)$ denotes the length of the loop γ.

Moreover these two functions are equivalent in a coarse sense.

1.2 Higher order Dehn functions

It is possible to extend the notion of Dehn functions to higher dimensions in the following way:

Definition 1.3. *(d-th order Dehn function)*

Let G be a group that acts on X by a geometric action; where X is a d-connected riemannian manifold or simplicial complex. We will define the d-th order Dehn function as a function that measures the volume necessary to extend a map $S^d \to X$ to a map $\overline{D}^{d+1} \to X$. More specifically if $f\colon S^d \to X$ is a Lipschitz map, we define:

$$\delta^d_X(f) = \inf \left\{ \text{vol}_{\,d+1}\, g |\ g\colon \overline{D}^{d+1} \to X,\ g\big|_{S^d} = f \right\}$$

and:

$$\delta^d_X(V) = \sup \left\{ \delta^d_X(f) |\ f\colon S^d \to X,\ \text{vol}_{\,d}(f) \leq V \right\}$$

We will call δ^d_X the d**th-order Dehn function** of G.

1.3 Filling Volume function

In a similar way as the last section, it is also possible to define a homological version of the Dehn function. We will denote $C^{lip}_d(X)$ as the set of **integral Lipschitz singular d-chains**, this is, the set of finite linear combinations, with integer coefficients, of Lipschitz maps from the Euclidean

d-dimensional simplex Δ^d to X. We will often call this simply a Lipschitz d-chain. Now by Rademacher's theorem we have that a Lipschitz map from the simplex to X is differentiable almost everywhere, and we can define the **volume** of the map as the integral of the magnitude of its jacobian. If a is a Lipschitz d-chain with $a = \sum n_i \alpha_i$; where n_i are integers and α_i are Lipschitz maps from Δ^d to X, we define the **mass** of a as:

$$\text{mass}_d(a) = \sum n_i \text{vol}_d(\alpha_i)$$

Definition 1.4. (*Filling volume function*) Let X be a d-connected riemannian manifold or a simplicial complex. Now let a be a integral Lipschitz d-cycle, we define the **filling volume** of a as:

$$FV_X^{d+1}(a) = \inf\{\text{mass } b | \ \partial b = a\}$$

where the infimum is taken over the set of all b in $C_d^{lip}(X)$ such that ∂b is a. Now, the filling volume function is:

$$FV_X^{d+1}(V) = \sup\{FV_X^{d+1}(a) \mid \text{mass}(a) \leq V\}$$

where this supremum is taken over all a integral Lipschitz d-cycles of mass$(a) \leq V$

The exact relationship between FV^{d+1} and δ^d depends on d. When $d \geq 3$, we have that $\delta_X^d \sim FV_X^{d+1}$ for all d-connected manifolds or simplicial complexes X. When $d = 2$, then $\delta_X^d \lesssim FV_X^{d+1}$ (see [7],App.2.(A'))

1.4 Carnot groups

The main theorem of this mémoire (Theorem 2.5) is focused on a certain special class of nilpotent Lie groups(Carnot groups) that allows us to extend certain argument in $\mathbb{R}^2$ about filling inequalities to these groups.

Definition 1.5. Let G be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, then lower central series:

$$\mathfrak{g} = \mathfrak{g}_0 \supset \cdots \supset \mathfrak{g}_{k-1} = 0, \ \ \mathfrak{g}_{i+1} = [\mathfrak{g}_i, \mathfrak{g}]$$

terminates. If $\mathfrak{g}_k = \{0\}$ and $\mathfrak{g}_{k-1} \neq \{0\}$, we say that $\mathfrak{g}$ has nilpotency class k. If there is a decomposition:

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_k$$

such that:

$$\mathfrak{g}_i = V_{i+1} \oplus \cdots \oplus V_k$$

and $[V_i, V_j] \subset V_{i+j}$, for all $i, j \leq k$, we call it a *grading* of $\mathfrak{g}$. If $\mathfrak{g}$ has a grading, we can extend the V_i to left invariant plane fields on G and give G a left-invariant metric such that the V_i's are orthogonal. With this metric, G is called a **Carnot group**.

If G is a Carnot group, there is a family of automorphisms $s_t \colon G \to G$ which acts on the Lie algebra by $s_t(v) = t^i v$ for all $v \in V_i$. These automorphisms distort vectors in $\mathfrak{g}$ by different amounts. Vectors in V_1 are distorted the least, and we call these vectors *horizontal*. If M is a manifold and $f \colon M \to G$ is a Lipschitz map, it is differentiable almost everywhere by Rademacher's theorem. If all of its tangent vectors lie in the plane field V_1, we say that f is *horizontal*; likewise is $\alpha \in C^{lip}(G)$, we say that α is horizontal is it is a sum of horizontal maps. If τ is a simplicial complex and $f \colon \tau \to G$ is a Lipschitz map which is horizontal on every simplex of τ of dimension at most k, we say that f is *k-horizontal*. We then have the following:

Lemma 1.6. Let $f \colon \Delta^k \to G$ be a horizontal map, then $\mathrm{vol}(s_t \circ f) = t^k \mathrm{vol}(f)$ for all $t \geq 0$

In later chapters of this mémoire, we will need the fact that given G a Carnot group, any Lipschitz function from G to G that has a bounded distance from the identity can be connected to the identity by a Lipschitz homotopy; in order to prove that, the following lemmas will be necessary to prove that statement:

Lemma 1.7. Given M, N two riemannian manifolds and $F \colon M \to N$ a differentiable function such that $||(TF)_p|| \leq C$ for all p in M, then we have that F is a C-Lipschitz function.

Proof. Let p, q be any two points of M. Let's consider a differentiable by parts path $\gamma \colon [0,1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Then as F is differentiable we also have that $F \circ \gamma$ is also differentiable by parts. We then have that:

$$d(F(p), F(q)) \leq L(F \circ \gamma)$$

Furthermore:

$$\begin{aligned} L(F\circ\gamma) &= \int_0^1 ||DF_{\gamma(t)}(\gamma'(t))||_{F(\gamma(t))}dt \\ &\le \int_0^1 C||(\gamma'(t))||_{\gamma(t)}dt \\ &\le CL(\gamma) \end{aligned}$$

Now as:

$$\begin{aligned} d(F(p),F(q)) &\le \inf_{\gamma(0)=p,\gamma(1)=q} L(F\circ\gamma) \\ &\le \inf_{\gamma(0)=p,\gamma(1)=q} C\cdot L(\gamma) \\ &\le C\cdot d(p,q). \end{aligned}$$

□

Lemma 1.8. Given G a Lie group with a left-invariant riemannian metric, $K\subset G$ a compact subset of G. Then there exists $C_K>0$ such that $R_g\colon G\to G$ given by $R_g(h)=h.g$ is C_K-Lipschitz for all g in K.

Proof. First of all, note that $L_h\colon G\to G$ given by $L_h(g)=h\cdot g$ is an isometry of G for all h in G. Secondly, note that $||(TR_g)_h||$ doesn't depend on h, because:

$$R_g\circ L_h = L_h\circ R_g$$

So, we know that:

$$\begin{aligned} ||(TR_g)_h|| &= \sup_{u\in\mathfrak{g}\setminus\{0\}} \frac{||TR_g(TL_hu)||}{||TL_hu||} \\ &= \sup_{u\in\mathfrak{g}\setminus\{0\}} \frac{||TL_h(TR_gu)||}{||u||} \\ &= ||(TR_g)_e|| \end{aligned}$$

Moreover:

$$\begin{aligned} ||(TR_g)_e|| &= \sup_{u\in\mathfrak{g}\setminus\{0\}} \frac{||TR_gu||}{||u||} \\ &= \sup_{u\in\mathfrak{g}\setminus\{0\}} \frac{||TL_{g^{-1}}(TR_gu)||}{||u||} \\ &= ||Ad(g^{-1})|| \end{aligned}$$

This clearly implies that there exists $C_K > 0$ such that $||TR_g|| \leq C_K$ for all g in K. Therefore by lemma 1.7 we have that R_g is C_K-Lipschitz for all g in K. □

Lemma 1.9. Given G a Carnot group, the automorphisms $s_t\colon G \to G$ are 1-Lipschitz for all $0 \leq t \leq 1$.

Proof. Clearly we only need to prove it for $t > 0$. In a similar manner as lemma 1.8 we have that:

$$s_t \circ L_g = L_{s_t(g)} \circ s_t$$

which implies that $||(Ts_t)_g||$ doesn't depend on g because:

$$\begin{aligned} ||(Ts_t)_g|| &= \sup_{u \in \mathfrak{g} \setminus \{0\}} \frac{||Ts_t(TL_g u)||}{||TL_g u||} \\ &= \sup_{u \in \mathfrak{g} \setminus \{0\}} \frac{||TL_{s_t(g)}(Ts_t u)||}{||u||} \\ &= \sup_{u \in \mathfrak{g} \setminus \{0\}} \frac{||Ts_t u||}{||u||} \\ &= ||(Ts_t)_e||. \end{aligned}$$

Now by definition we know that $Ts_t(u) = t^i u$ for all u in V_i which clearly implies that $||(Ts_t)_e|| \leq t \leq 1$ which by lemma 1.7 implies that s_t is 1-Lipschitz for all t in $[0, 1]$. □

Lemma 1.10. Given G a Carnot group, and $\phi\colon G \to G$ a Lipschitz function such that $d(g, \phi(g)) < c$ for all g in G. Then we can find a Lipschitz homotopy $H\colon G \times [0, 1] \to G$ such that:

$$\begin{aligned} H(\cdot, 0) &= id \\ H(\cdot, 1) &= \phi \end{aligned}$$

Proof. As $d(g, \phi(g)) \leq c$ for all g in G, there exists $\overline{\phi}\colon G \to G$ such that:

$$\phi(g) = g \cdot \overline{\phi}(g)$$

and that $\overline{\phi}(g)$ belongs to $\overline{B}(e, c)$ for all g in G. Let's denote C_K as a constant that comes from Lemma 1.8 for $K = \overline{B}(e, c)$ We will now define the homotopy H as:

$$H(g, t) = g \cdot s_t(\overline{\phi}(g))$$

Now, in order to prove that H is Lipschitz we will first prove that $\overline{\phi}$ is Lipschitz. Let's consider g, h any two elements of G. Then we have:

$$\begin{aligned} d\left(\overline{\phi}(g), \overline{\phi}(h)\right) &= d\left(g^{-1} \cdot \phi(g), h^{-1} \cdot \phi(h)\right) \\ &\leq d\left(g^{-1} \cdot \phi(g), g^{-1} \cdot \phi(h)\right) + d\left(g^{-1} \cdot \phi(h), h^{-1} \cdot \phi(h)\right) \\ &\leq \text{Lip}(\phi) d(g,h) + d\left(h \cdot \overline{\phi}(h), g \cdot \overline{\phi}(h)\right) \\ &\leq (\text{Lip}(\phi) + C_K) d(g,h). \end{aligned}$$

Therefore, we have that:

$$\begin{aligned} d\left(H(g,t), H(h,t)\right) &= d\left(g \cdot s_t(\overline{\phi}(g)), h \cdot s_t(\overline{\phi}(h))\right) \\ &\leq d\left(g \cdot s_t(\overline{\phi}(g)), h \cdot s_t(\overline{\phi}(g))\right) + d\left(h \cdot s_t(\overline{\phi}(g)), h \cdot s_t(\overline{\phi}(h))\right) \\ &\leq C_K d(g,h) + \text{Lip}(\overline{\phi}) d(g,h) = \left(C_K + \text{Lip}(\overline{\phi})\right) d(g,h). \end{aligned}$$

Moreover, let's consider $I \colon K \times [0,1] \to K$ given by:

$$I(g,t) = s_t(g)$$

there exists a constant C'_K such that for all g in K and t in $[0,1]$, we have that:

$$||(TI)_{(g,t)}|| \leq C'_K$$

Which implies that:

$$\begin{aligned} d(H(g,t), H(g,s)) &= d(g \cdot I\left(\overline{\phi}(g), t), g \cdot I(\overline{\phi}(g), s)\right) \\ &= d(I(\overline{\phi}(g), t), I(\overline{\phi}(g), s)) \\ &\leq c'_K |s - t|. \end{aligned}$$

Now for all $g, h \in G$ and $s, t \in [0,1]$:

$$\begin{aligned} d(H(g,t), H(h,s)) &\leq d(H(g,t), H(g,s)) + d(H(g,s), H(h,s)) \\ &\leq C'_K |s-t| + (Lip(\overline{\phi}) + C_K) d(g,h) \\ &\leq \max\{C'_K, Lip(\overline{\phi}) + C_K\} \left(|s-t| + d(g,h)\right) \end{aligned}$$

which implies that H is a Lipschitz homotopy.

□

1.5 The Federer-Fleming Deformation theorem

A key tool to work with Lipschitz chains is the Federer-Fleming Deformation Theorem, which states that Lipschitz chains can be approximated by simplicial chains.

A triangulation of a space X consists of a simplicial complex τ and a homeomorphism $f\colon \tau \to X$. We will sometimes refer to the tuple (τ, f) just as τ, when it is implicit. We can put a metric on τ so that each simplex is isometric to the standard Euclidean simplex(we can embed it in a standard simplex and then consider the induced metric) ; if f is a biLipschitz map(f and f^{-1} are both Lipschitz); then we call (τ, f) a Lipschitz triangulation. From now on, we will assume all triangulations to be Lipschitz.

If $(\tau, f\colon \tau \to X)$ is a triangulation of X, then a simplicial k-chain of τ is a formal sum of k-dimensional faces of τ. We can use the push-forward map f to identify such chains with Lipschitz chains of X. By abuse of notation, we call these chains simplicial chains in X, and we denote the complex of simplicial chains by $C_*(\tau)$. Federer and Fleming showed that Lipschitz chains can be approximated by simplicial chains [1]. The version we state is a slight simplification; it applies only to Lipschitz chains with simplicial boundaries. In this theorem, $P_\tau(\alpha)$ will be a simplicial approximation of and $Q_\tau(\alpha)$ will be a Lipschitz chain which interpolates between α and $P_\tau(\alpha)$.

Theorem 1.11. (Deformation Theorem) Let $(\tau, f\colon \tau \to X)$ be a triangulation of X. There is a constant $c = c(\tau)$ such that if $\alpha \in C_k^{lip}(X)$ is a chain such that $\partial\alpha \in C_{k-1}(\tau)$, then there are $P_\tau(\alpha) \in C_k(\tau)$ and $Q_\tau(\alpha) \in C_{k+1}^{lip}(X)$ such that:

1. mass $P_\tau(\alpha) \leq c.$ mass(α)
2. mass $Q_\tau(\alpha) \leq c.$ mass(α), and
3. $\partial Q_\tau(\alpha) = \alpha - P_\tau(\alpha)$.

Federer and Fleming originally proved their theorem in the case of Lipschitz currents in $\mathbb{R}^n$(cf. [1]); so this statement is somewhat different from their original, it is closest to the version proved in [2].

It can also be seen from the proof in [2] that there is homotopical version of Theorem 1.11 which we will use also later in our proof of Theorem 2.10.

Definition 1.12. (Brady,Bridson,Forester,Shankar[11])

Given M a compact k-dimensional manifold, τ a simplicial complex, and a Lipschitz map $\alpha\colon M \to \tau$, we say that $\alpha\colon M \to X$ is admissible if the image of α lies on the (k)-skeleton $\tau^{(k)}$ of τ, and if $\alpha^{-1}(\tau^{(k)} \setminus \tau^{(k-1)})$ is a disjoint union of open k-balls, each mapped homeomorphically onto a k-cell of τ. If $(\tau, f\colon \tau \to N)$, is a triangulation of N, we say that $\alpha\colon M \to N$ is τ-admissible if and only if $f^{-1} \circ \alpha$ is an admissible map to τ.

It can also be seen that a simplicial map between simplicial complexes is admissible.

Furthermore, one can then prove the following:

Theorem 1.13. *(Homotopic Deformation Theorem)* Let $(\tau, f\colon \tau \to X)$ be a triangulation of X and let M be a compact k-manifold. There is a constant $c = c(\tau)$ such that if $\alpha\colon M \to X$ is a Lipschitz map and $\alpha\big|_{\partial M}$ is τ-admissible, then there is a τ-admissible map $P'_\tau(\alpha)\colon M \to X$ and a Lipschitz homotopy $Q'_\tau(\alpha)\colon M \times [0,1] \to X$ such that:

1. vol $P'_\tau(\alpha) \leq c \cdot \mathrm{vol}\, \alpha$,

2. vol $Q'_\tau(\alpha) \leq c \cdot \mathrm{vol}\, \alpha$, and

3. $Q'_\tau(\alpha)$ is a homotopy between α and $P'_\tau(\alpha)$ which is constant on ∂M.

Chapter 2

Filling cycles and spheres through approximations

2.1 Main Idea

The basic idea is the following. Given α a curve of length l in $\mathbb{R}^2$. We can approximate α in successively larger grids: Let $P_0(\alpha)$ be the approximation of α in a 1×1 grid, $P_1(\alpha)$ the approximation of α in a 2×2 grid, and so on until $i \ll \log_2 l$. We will assume that length$(P_i(\alpha)) \approx l$ (to formalize this we need the Federer-Fleming theorem). Moreover when $i \gg \log_2 l$ we can approximate α by the zero cycle. We can connect approximations using annuli made of squares, and since $P_i(\alpha)$ and $P_{i+1}(\alpha)$ are close together, it takes relatively few such squares; just as $P_i(\alpha)$ is made up of $\approx l2^{-i}$ segments of length 2^i, $R_i(\alpha)$ is a sum of $\approx l2^{-i}$ squares with side 2^i, and thus area $\approx l2^i$. If i_0 is such that $2^{i_0} \gg l$, then $P_{i_0}(\alpha) = 0$, so we get a filling of α by taking the sum of the $R_i(\alpha)$'s, we let:

$$\beta = \sum_{i=0}^{i_0} R_i(\alpha)$$

then $\partial\beta = \alpha$ and the area of β is $\approx l^2$

A similar argument can be used to fill higher-dimensional cycles in higher dimensional Euclidean spaces: if α is a k-cycle of mass V in $\mathbb{R}^n$, it can be approximated by a cycle $P_i(\alpha)$ of $\approx V/2^{ik}$ k-cubes of side-length 2^i by using a $2^i \times \ldots \times 2^i$ grid in $\mathbb{R}^n$. Moreover, $P_i(\alpha)$ and $P_{i+1}(\alpha)$ can be connected by a chain $R_i(\alpha)$ consisting of $\approx V/2^{ik}$ $(k+1)$-cubes of side-length 2^i. As before, if i_0 is such that $2^{i_0 k} \gg V$, then α is smaller than any individual cube, so $P_{i_0}(\alpha) = 0$ and if we let:

$$\beta = \sum_{i=0}^{i_0} R_i(\alpha)$$

The key step in extending this argument to some Carnot groups is to construct the $P_i(\alpha)$ and $R_i(\alpha)$; that is, to construct a sequences of coarser and coarser approximations by chains and then connect them by chains.

The bounds we obtain depend on how efficiently we can produce simplicial approximations. The mass of a simplicial approximation is controlled by a constant factor c in the Federer-Fleming Deformation Theorem (Theorem 1.11), which depends on the triangulation used. In $\mathbb{R}^n$, scaling maps allowed us to construct a family of triangulations with different sized simplices and the same c but this is not always possible in a nilpotent group. In the rest of the section we will prove that if G is a Carnot group and that "there exists sufficiently many horizontal maps into G", then we can produce efficient simplicial approximations at all scales and use these to find strong bounds on the Filling Volume and higher order Dehn functions of G.

2.2 Simplicial Approximation in Carnot groups

We will construct $P_i(\alpha)$ and $R_i(\alpha)$ using the Federer-Fleming Deformation theorem (Theorem 1.11). This allows us to approximate Lipschitz chains and cycles in G by simplicial chains and cycles in a triangulation of G. When G is Carnot, we can construct triangulations of G by scaling a single triangulation τ; this gives triangulations with simplices of different scales, and approximations in these triangulations give the P_i and R_i.

One difficulty is that the scaling automorphism may distort these triangulations. The scaling automorphism $s_t\colon \mathbb{R}^n \to \mathbb{R}^n$ stretches each direction by a factor of t. In a Carnot group, however, the scaling automorphism $s_t\colon G \to G$ may scale vectors by up to t^c, where c is the nilpotency class of G. Since the P_i and R_i are made up of scaled simplices, a bad choice of τ may lead to very bad approximations. To avoid this we will use k-horizontal triangulations. Specifically, we will show that if certain

k-horizontal maps and triangulations exist, then $P_i(\alpha)$ can be constructed so that mass $P_i(\alpha) \leq$ mass α for all i, and that this leads to bounds on the filling volume function. (In the next chapter, we will construct some groups which have such triangulations)

In the rest of this section, G will represent a Carnot group and Γ a lattice in G. The maps $s_t \colon G \to G$, $t \geq 0$, will represent the family of scaling automorphisms of G.

Let $(\tau, f \colon \tau \to G)$ be a triangulation of G and let P_τ and Q_τ be as in Theorem 1.11. If f is k-horizontal and α is a Lipschitz k-cycle, then $P(\alpha)$ is a horizontal approximation of α: Unfortunately; constructing these horizontal triangulations can be difficult in most cases, instead we will construct a slightly different approximation. Let $(\tau, f \colon \tau \to G)$ be a triangulation of G, with no conditions on the horizontality of f. Let $\phi \colon \tau \to G$ be a k-horizontal Lipschitz map which is a bounded distance from f; i.e., there is a c such that $d(f(x), \phi(x)) < c$ for all x in τ. If α is a Lipschitz k-cycle in G, then $f_\#^{-1}(\alpha)$ is a Lipschitz k-cycle in τ, and $P_\tau(f_\#^{-1}(\alpha))$ is a simplicial cycle in τ. By abuse of notation we define $P_{\phi(\tau)}(\alpha)$ by:

$$P_{\phi(\tau)}(\alpha) = \phi_\# \left[P_\tau(f_\#^{-1}(\alpha)) \right]$$

This is a sum of images of k-simplices of τ, so it is a horizontal cycle.

We will show that the cycle $P_{\phi(\tau)}(\alpha)$ is close to α in the sense that there is a chain $Q_{\phi(\tau)}(\alpha)$ with mass comparable to the mass of α which interpolates between α and $P_{\phi(\tau)(\alpha)}$.

Lemma 2.1. There is a c_Q depending only on ϕ, τ and k such that for all k-cycles α there is a $(k+1)$-chain $Q_{\phi(\tau)}(\alpha)$ such that:

$$\partial Q_{\phi(\tau)}(\alpha) = P_{\phi(\tau)}(\alpha) - \alpha$$

and mass $Q_{\phi(\tau)}(\alpha) \leq c_Q$ mass α

Proof. Because $\phi \circ f^{-1}$ moves each point of G by a bounded distance, by the lemma 1.10 there is a Lipschitz homotopy $h \colon G \times [0,1] \to G$ between id_G and $\phi \circ f^{-1}$. We will define:

$$Q_{\phi(\tau)}(\alpha) = h_\#(\alpha \times [0,1]) + \phi_\#[Q_\tau(f_\#^{-1}(\alpha))]$$

For the first part of the lemma, note that:

$$\begin{aligned}\partial Q_{\phi(\tau)}(\alpha) &= h_\#(\partial(\alpha \times [0,1])) + \phi_\#[\partial Q_\tau(f_\#^{-1}(\alpha))] \\ &= \phi_\#(f_\#^{-1}(\alpha)) - \alpha + \phi_\#[P_\tau(f_\#^{-1}(\alpha))] - \phi_\#(f_\#^{-1}(\alpha)) \\ &= \phi_\#[P_\tau(f_\#^{-1}(\alpha))] - \alpha \\ &= P_{\phi(\tau)}(\alpha) - \alpha\end{aligned}$$

For the second part, note that there is a c such that:

$$\text{mass } Q_\tau(f_\#^{-1}(\alpha)) \leq c \text{ mass} f_\#^{-1}(\alpha)$$

If we let:

$$c_Q = \text{Lip}(h)^{k+1} + \text{Lip}(f^{-1})^k \text{Lip}(\phi)^{k+1} c$$

The second part of the lemma is straightforward.

□

We can compose $P_{\phi(\tau)}$ with scaling automorphisms to produce a sequence of approximations. To avoid cumbersome subscripts, we will abuse notation by writing $s_t(\alpha)$ instead of $(s_t)_\#(\alpha)$ when intention is clear. We define:

$$P_i(\alpha) = s_{2^i}(P_{\phi(\tau)(s_{2^{-i}}(\alpha))})$$

this is a horizontal approximation of α with simplices of diameter $\sim 2^i$

Lemma 2.2. There is a c_P depending only on ϕ and τ such that for all $i \geq 0$ and for all k-cycles α, we have mass $P_i(\alpha) \leq c_P$.mass α

Proof. Note that by the choice of the metric on G, for any k-chain σ and any $0 \leq t \leq 1$:

$$\text{mass } s_t(\sigma) \leq t^k \text{mass } \sigma$$

and that for any horizontal k-chain σ; in particular when $\sigma = P_{\phi(\tau)}(s_{2^{-i}}(\alpha))$ and any $t > 0$:

$$\text{mass } s_t(\sigma) = t^k \text{mass } \sigma$$

By the theorem of Federer and Fleming, there is a c such that:

$$\text{mass } P_\tau(\sigma) \leq c \text{ mass } \sigma$$

for all k-chains σ. If we let $c_P = \text{Lip}(f^{-1})^k \text{Lip}(\phi)^k c$, then:

$$\text{mass } P_i(\alpha) \leq 2^{ik} \text{Lip}(f^{-1})^k \text{Lip}(\phi)^k c 2^{-ik} \text{mass } \alpha \leq c_P \text{mass } \alpha$$

as desired. □

Next, we construct simplicial chains interpolating between two different approximations of a cycle. The basic idea is that if $(\tau_0, f_0\colon \tau_0 \to G)$ and $(\tau_1, f_1\colon \tau_1 \to G)$ are two triangulations of G, we can connect approximations in τ_0 and τ_1 using a triangulation of $G \times [0,1]$ which interpolates between τ_0 and τ_1.

Let $(\eta, g\colon \eta \to G \times [0,1])$ be a triangulation of $G \times [0,1]$. For $i = 0, 1$, suppose that $g^{-1}(G \times \{i\})$ is a subcomplex of η which is isomorphic to τ_i under an isomorphism $\iota_i\colon \tau_i \cong g^{-1}(G \times \{i\})$ such that $g \circ \iota_i = f_i$. We say that η restricts to τ_i on $G \times \{i\}$. Let $\psi\colon \eta \to G$ be a $(k+1)$-horizontal map and let $\phi_i\colon \tau_i \to G$ be defined by $\phi_i = \psi \circ \iota_i$ for $i = 0, 1$. We will construct a horizontal $(k+1)$-chain interpolating between $P_{\phi_0(\tau_0)}(\alpha)$ and $P_{\phi_1(\tau_1)}(\alpha)$.

Lemma 2.3. There is a c_R depending only on ψ and η such that for all $i \geq 0$ and for all k-cycles α, there is a $(k+1)$-chain $R_{\psi(\eta)(\alpha)}$ such that:

$$\partial R_{\psi(\eta)}(\alpha) = P_{\phi_1(\tau_1)}(\alpha) - P_{\phi_0(\tau_0)}(\alpha)$$

and :

$$\text{mass } R_{\psi(\eta)}(\alpha) \leq c_R \text{ mass } \alpha$$

Proof. Define:

$$X(\alpha) = Q_{\tau_1}[(f_1^{-1})_\#(\alpha)] + g_\#^{-1}(\alpha \times [0,1]) - Q_{\tau_0}[(f_0^{-1})_\#(\alpha)]$$

Note that:

$$\begin{aligned}\partial X(\alpha) &= P_{\tau_1}[(f_1^{-1})_\#(\alpha)] - (f_1^{-1})_\#(\alpha) + g_\#^{-1}(\alpha \times \{1\} - \alpha \times \{0\}) \\ &\quad - P_{\tau_0}[(f_0^{-1})_\#(\alpha)] + (f_0^{-1})_\#(\alpha) \\ &= P_{\tau_1}[(f_1^{-1})_\#(\alpha)] - P_{\tau_0}[(f_0^{-1})_\#(\alpha)]\end{aligned}$$

Now define:

$$R_{\psi(\eta)}(\alpha) = \psi_\# \left(P_\eta(X(\alpha))\right)$$

This is the image of a simplicial chain, and:

$$\partial R_{\psi(\eta)}(\alpha) = \psi_\# \left(\partial X(\alpha)\right) = P_{\phi_1(\tau_1)}(\alpha) - P_{\phi_0(\tau_0)}(\alpha)$$

as desired. For the bound on the mass of $R_{\psi(\eta)}(\alpha)$, note that by theorem 1.11, there is a c such that:

$$\text{mass } P_\tau(\sigma) \leq c \text{ mass } \sigma$$

and:

$$\text{mass } Q_\tau(\sigma) \le c \text{ mass } \sigma$$

for all k-chains or $k+1$-chains σ. Thus:

$$\text{mass } X(\alpha) \le \left[2c(\text{Lip } g^{-1})^k + (\text{Lip } g^{-1})^{k+1}\right] \text{mass } \alpha$$

If we let:

$$c_R = c\left[2c(\text{Lip } g^{-1})^k + (\text{Lip } g^{-1})^{k+1}\right] \text{Lip}(\psi)^{k+1}$$

then we have:

$$\text{mass } R_{\psi(\eta)}(\alpha) \le c_R \text{mass } \alpha$$

□

In this case we would like to connect $P_0(\alpha) = P_{\psi(\eta)}(\alpha)$ and $P_1(\alpha) = s_2(P_{\phi(\tau)}(s_{2^{-1}}(\alpha)))$. Let $(\tau_0, f_0\colon \tau_0 \to G) = (\tau, f)$ and $\phi_0 = \phi$ and define $(\tau_1, f_1\colon \tau_1 \to G)$ by letting $\tau_1 = \tau$ and $f_1 = s_2 \circ f$. If we define $\phi_1 = s_2 \circ \phi$, then we have $P_1(\alpha) = P_{\phi_1(\tau_1)}(\alpha)$. We will define $R_0(\alpha)$ as $R_{\psi(\eta)}(\alpha)$ for an appropiate ψ and η and obtain R_i by conjugating R_0 by s_{2^i}.

Lemma 2.4. Let $k > 0$. Let $(\tau, f\colon \tau \to G)$ be a triangulation and let $\phi\colon \tau \to G$ be a $(k+1)$-horizontal map which is a bounded distance from f. Define $\tau_0, \phi_0, \tau_1, \phi_1$ as above. Let $(\eta, g\colon \eta \to G \times [0,1])$ which restricts to τ_i on $G \times \{i\}$, $i = 0, 1$. Let $\iota_i\colon \tau_i \cong g^{-1}(G \times \{i\})$ be the implied isomorphism. Let $\psi\colon \eta \to G$ be a $(k+1)$-horizontal map which extends the ϕ_i. If we define:

$$R_i(\alpha) = s_{2^i}(R_{\psi(\eta)}(s_{2^{-i}}(\alpha)))$$

then for all i and for all k-cycles α we have that:

$$\partial R_i(\alpha) = P_{i+1}(\alpha) - P_i(\alpha)$$

and:

$$\text{mass } \ R_i(\alpha) \le c_R 2^i \text{mass } \ \alpha$$

where c_R is the constant from lemma 2.3 corresponding to ψ and η.

Proof. It follows from lemma 2.3 that:

$$\begin{aligned}\partial R_i(\alpha) &= s_{2^i}\left[P_{\phi_1(\tau_1)}(s_{2^{-i}}(\alpha)) - P_{\phi_0(\tau_0)}(s_{2^{-i}}(\alpha))\right] \\ &= (s_{2^i} \circ P_{\phi_1(\tau_1)} \circ s_{2^{-i}})(\alpha) - (s_{2^i} \circ P_{\phi_0(\tau_0)} \circ s_{2^{-i}})(\alpha)\end{aligned}$$

we have that $\partial R_i(\alpha)$ as desired. Next we bound the mass of $R_i(\alpha)$. Recall that mass $R_0(\alpha) \leq c_R$ mass α for all α. Then:

$$\text{mass } R_i(\alpha) = \text{mass } s_{2^i}(R_i(s_{2^{-i}}(\alpha)))$$

since $R_i(s_{2^{-i}}(\alpha))$ is a horizontal $(k+1)$-cycle, we have that:

$$\text{mass } R_i(\alpha) \leq 2^{(k+1)i} c_R 2^{-ik} \text{mass } \alpha = c_R 2^i \text{mass } \alpha.$$

as desired. □

when such an R_i exists, we can use it to prove filling inequalities.

Theorem 2.5. (R.Young) Given G a Carnot group, $k, (\tau, f), \phi, (\eta, g)$ and ψ that satisfy the hypotheses of lemma 2.4. Then:

$$FV_G^{k+1}(V) \preceq V^{\frac{k+1}{k}}$$

Proof. It is enough to show that there is a c such that if α is a k-cycle with sufficiently large volume, then there is a chain β such that $\partial\beta = \alpha$ and

$$\text{mass } \beta \leq c(\text{mass } \alpha)^{\frac{k+1}{k}}$$

First, we claim that when i is large, then $P_i(\alpha) = 0$. Let :

$$c' = c_\tau \text{ Lip}(f^{-1})^k \text{vol } (\Delta^k)^{-1},$$

where c_τ is the constant from Theorem 1.11 and Δ^k is the standard Euclidean k-simplex. Let i_0 be the integer such that:

$$2^{(i_0-1)k} \leq c' \text{mass } \alpha < 2^{i_0 k},$$

and suppose that $i \geq i_0$. Let $X = (P_\tau \circ f_\#^{-1} \circ s_{2^{-i}})(\alpha)$ so that $P_i(\alpha) = s_{2^i}(\phi_\#(X))$. We claim that $X = 0$. Since X is an integral simplicial k-cycle, it suffices to show that mass $X <$ vol Δ^k. But by theorem 1.11, we have that :

$$\text{mass } X \leq c_\tau Lip(f^{-1})^k 2^{-ki} \text{mass } \alpha < \text{vol } \Delta^k.$$

So $X = 0$ and thus $P_i(\alpha) = 0$. We claim that:

$$\beta = -Q_{\phi(\tau)}(\alpha) + \sum_{i=0}^{i_0-1} R_i(\alpha)$$

is a filling of α with mass $\preceq (\text{mass }\alpha)^{\frac{k+1}{k}}$. First, note that:

$$\partial\beta = \alpha - P_{i_0}(\alpha) = \alpha$$

Furthemore, by the previous lemmas, we have that:

$$\begin{aligned}\text{mass }\beta &\leq c_Q\text{mass }\alpha + \sum_{i=0}^{i_0-1} c_R 2^i \text{mass }\alpha \\ &\leq (c_Q + c_R 2^{i_0})\text{mass }\alpha \\ &\leq (c_Q + 2c_R(c'\text{mass }\alpha)^{\frac{1}{k}})\text{mass }\alpha\end{aligned}$$

For mass α sufficiently large, we have:

$$\text{mass }\beta \leq 4c_R(c'\text{mass }\alpha)^{\frac{k+1}{k}}$$

as desired. □

2.3 Homotopic Filling bounds

In this section we will give a sketch of how to adapt these arguments to produce fillings of spheres by balls rather than fillings of chains by cycles. The main change in the arguments is that now we are going to use a homotopical version of the Federer Fleming Deformation theorem(theorem 1.13).

We will start then with a family of lemmas which are homotopical versions of the lemmas 2.1-2.4.

Following this line of thought, in a similar manner as lemma 2.1 we will denote as $P_{\phi(\tau)}(\alpha)\colon S^k \to G$ the Lipschitz map given by:

$$P_{\phi(\tau)}(\alpha) = \phi \circ P_\tau(f^{-1} \circ \alpha)$$

Lemma 2.6. There is a c_Q depending only on ϕ, τ and k such that for all $\alpha\colon D^k \to G$ Lipschitz map, there is an homotopy $Q_{\phi(\tau)}(\alpha)\colon S^k \times [0,1] \to G$ such that:

- $Q_{\phi(\tau)}(\alpha)(\cdot, 0) = P_{\phi(\tau)}(\alpha)(\cdot)$ and
- $Q_{\phi(\tau)}(\alpha)(\cdot, 1) = \alpha(\cdot)$.

and vol $Q_{\phi(\tau)}(\alpha) \leq c_Q$ vol α.

Proof. By lemma 1.10, there is a Lipschitz homotopy $h\colon G \times [0,1] \to G$ between id_G and $\phi \circ f^{-1}$. Let's define now:

$$Q_{\phi(\tau)}(\alpha)(\cdot,t) = \begin{cases} h(\alpha(\cdot),2t) & \text{for all } t \in [0,\frac{1}{2}] \\ \phi(Q_\tau(f^{-1}\circ\alpha)(\cdot,2t-1)) & \text{for all } t \in [\frac{1}{2},1]. \end{cases}$$

Clearly $Q_{\phi(\tau)}(\alpha)$ is a Lipschitz homotopy between α and $P_{\phi(\tau)}(\alpha)$. Moreover if we let:

$$c_Q = (\text{Lip}h)^{k+1} + \text{Lip}(\phi)^{k+1}\text{Lip}(f^{-1})^k$$

We have that:

$$\begin{aligned} \text{vol }(Q_{\phi(\tau)}(\alpha)) &\leq \text{Lip}(h)^{k+1}\text{vol }\alpha + \text{Lip}(\phi)^{k+1}\text{vol }(f^{-1}\circ\alpha) \\ &\leq \text{Lip}(h)^{k+1}\text{vol }\alpha + \text{Lip}(\phi)^{k+1}\text{Lip}(f^{-1})^k\text{vol }\alpha \\ &\leq c_Q\text{vol }\alpha \end{aligned}$$

Which is what we wanted to prove. □

Lemma 2.7. We can compose $P_{\phi(\tau)}$ with the scaling automorphisms in order to get the good "approximations" of α in the respective dilated triangulations, more specifically we can define:

$$P_i(\alpha) = s_{2^i} \circ (P_{\phi(\tau)}(s_{2^{-i}} \circ \alpha))$$

It is clear that $P_i(\alpha)$ is an horizontal approximation of α in the $(\tau, s_{2^i}\circ f)$ triangulation. Furthermore, there is a c_P depending only on ϕ and τ such that for all $i \geq 0$ and for all Lipschitz maps $\alpha\colon S^k \to G$ we have that:

$$\text{vol } P_i(\alpha) \leq c_P\text{vol }\alpha$$

Proof. Note that by the choice of the metric on G for any Lipschitz map $\sigma\colon S^k \to G$ and any t in $[0,1]$ we have that:

$$\text{vol } s_t(\sigma) \leq t^k\text{vol }\sigma$$

and that any horizontal Lipschitz map σ (in particular when $\sigma = P_{\phi(\tau)}(s_{2^{-i}}(\alpha))$) and any $t > 0$:

$$\text{vol } s_t(\sigma) = t^k\text{vol }\sigma$$

By theorem 1.13 there is a c such that:

$$\text{vol } P_\tau(\sigma) \leq c\text{vol } \sigma$$

If we let $c_P = \text{Lip}(f^{-1})^k \text{Lip}(\phi)^k c$, then we have that:

$$\text{vol } P_i(\alpha) \leq 2^{ik} \text{Lip}(f^{-1})^k \text{Lip}(\phi)^k c 2^{-ik} \text{vol } \alpha$$

as desired. ∎

Next we will construct adequate homotopies between two different approximations of a Lipschitz map $\alpha \colon S^k \to X$. The basic idea is that if $(\tau_0, f_0 \colon \tau_0 \to G)$ and $(\tau_1, f_1 \colon \tau_1 \to G)$ are two triangulations of G, we can connect approximations in τ_0 and τ_1 using a triangulation of $G \times [0,1]$ which interpolates between τ_0 and τ_1.

Let $(\eta, g \colon \eta \to G \times [0,1])$. For $i = 0, 1$, suppose that $g^{-1}(G \times \{i\})$ is a subcomplex of η which is isomorphic to τ_i under an isomorphism $\iota_i \colon \tau_i \cong g^{-1}(G \times \{i\})$ such that $g \circ \iota_i = f_i$. We say that η restricts to τ_i on $G \times \{i\}$. Let $\psi \colon \eta \to G$ be a $(k+1)$-horizontal map and let $\phi_i \colon \tau_i \to G$ be defined by $\phi_i \colon \tau_i \to G$ be defined by $\phi_i = \psi \circ \iota_i$ for $i = 0, 1$. We will construct a horizontal homotopy between $P_{\phi_0(\tau_0)}(\alpha)$ and $P_{\phi_1(\tau_1)}(\alpha)$.

Lemma 2.8. There is a c_R depending only on ψ and η such that for all $i \geq 0$ and for all Lipschitz maps $\alpha \colon S^k \to G$, there is a Lipschitz homotopy $R_{\psi(\eta)}(\alpha)$ between $P_{\phi_1(\tau_1)}(\alpha)$ and $P_{\phi_0(\tau_0)}(\alpha)$ such that:

$$\text{vol } R_{\psi(\eta)}(\alpha) \leq c_R \text{vol } \alpha$$

Furthermore $R_{\psi(\eta)}(\alpha)$ is horizontal.

Proof. Let's define:

$$X(\alpha)(\cdot, t) = \begin{cases} Q_{\tau_1}(f_1^{-1} \circ \alpha) & \text{for all } t \in [0, \frac{1}{3}] \\ g^{-1}(\alpha(\cdot), 3t - 1) & \text{for all } t \in [\frac{1}{3}, \frac{2}{3}] \\ Q_{\tau_0}(f_0^{-1} \circ \alpha) & \text{for all } t \in [\frac{2}{3}, 1] \end{cases}$$

Furthermore, consider:

$$R_{\psi(\eta)}(\alpha) = \psi \circ (P_\eta(X(\alpha)))$$

clearly this is a homotopy between $P_{\phi_1(\tau_1)}(\alpha)$ and $P_{\phi_0(\tau_0)}(\alpha)$ as desired. For the bound on the volumen of $R_{\psi(\eta)}(\alpha)$ there is a c such that:

$$\text{vol } P_\tau(\sigma) \leq c\text{vol } \sigma$$

and

$$\text{vol } Q_\tau(\sigma) \leq c\text{vol } \sigma$$

for all $\sigma \colon S^k \to G$ and for all $\sigma \colon S^k \times [0,1] \to G$ where σ is a Lipschtiz map. Thus:

$$\text{vol } X(\alpha) \leq [2c(\text{Lip}g^{-1})^k + (\text{Lip}g^{-1})^{k+1}]\text{vol } \alpha$$

If we let:

$$c_R = c\left[2c(\text{Lip}g^{-1})^k + (\text{Lip}g^{-1})^{k+1}\right]\text{Lip}(\psi)^{k+1}.$$

then we have that:

$$\text{vol } R_{\psi(\eta)}(\alpha) \leq c_R\text{vol } \alpha$$

as desired. □

Now let's define $(\tau_0, f_0 \colon \tau_0 \to G) = (\tau, f)$ and $\phi_0 = \phi$ and define $(\tau_1, f_1 \colon \tau_1 \to G)$ by letting $\tau_1 = \tau$ and $f_1 = s_2 \circ f$. If we define $\phi_1 = s_2 \circ \phi$, then we have $P_1(\alpha) = P_{\phi_1(\tau_1)}(\alpha)$. We will define $R_0(\alpha)$ as $R_{\psi(\eta)}(\alpha)$ for an appropiate ψ and η and obtain R_i by conjugating R_0 by s_{2^i}. More specifically:

Lemma 2.9. Given $k > 0$, $(\tau, f \colon \tau \to G)$ a triangulation of G and $\phi \colon \tau \to G$ be a $(k+1)$-horizontal map which is a bounded distance from f. Define $\tau_0, \phi_0, \tau_1, \phi_1$ as above.

Let $(\eta, g \colon \eta \to G \times [0,1])$ be a triangulation of $G \times [0,1]$ which restricts to τ_i on $G \times \{i\}$ on $G \times \{i\}$, $i = 0, 1$. Let $\iota_i \colon \tau_i \cong g^{-1}(G \times \{i\})$ be the implied isomorphism. Let $\psi \colon \eta \to G$ be a $(k+1)$-horizontal map which extends the ϕ_i(i.e., $\phi_i = \psi \circ \iota_i,\ i = 0, 1$). If we define:

$$R_i(\alpha) = s_{2^i} \circ (R_{\psi(\eta)}(s_{2^{-i}} \circ \alpha))$$

then for all $i \geq 0$ and for all $\alpha \colon S^k \to G$ Lipschitz maps we have that $R_i(\alpha)$ is a Lipschitz homotopy between $P_{i+1}(\alpha)$ and $P_i(\alpha)$. Moreover we have that:

$$\text{vol } R_i(\alpha) \leq c_R 2^i \text{vol } \alpha$$

Where c_R is the constant from lemma 2.8 corresponding to ψ and η.

Proof. It follows from lemma 2.8 that $R_i(\alpha)$ is an homotopy between $s_{2^i} \circ P_{\phi_0(\tau_0)}(s_{2^{-i}} \circ \alpha)$ and $s_{2^i} \circ P_{\phi_1(\tau_1)}(s_{2^{-i}} \circ \alpha)$ which by definition are $P_i(\alpha)$ and $P_{i+1}(\alpha)$ respectively. Furthermore, we have that:

$$\begin{aligned}\text{vol } R_i(\alpha) &= \text{vol } s_{2^i} \circ R_0(s_{2^{-i}} \circ \alpha) \\ &\leq 2^{(k+1)i} c_R \text{vol } (R_0(s_{2^{-i}} \circ \alpha))\end{aligned}$$

Since $R_0(s_{2^{-i}} \circ \alpha)$ is horizontal we have that:

$$\text{vol } R_i(\alpha) \leq 2^{(k+1)i} c_R 2^{-ik} \text{vol } \alpha = c_R 2^i \text{vol } \alpha$$

as desired. □

This can be used to prove:

Theorem 2.10. (R.Young) Given $k, (\tau, f), \phi, (\eta, g)$ and ψ that satisfy the hypotheses of lemma 2.4. Then we have that:

$$\delta^k_G(V) \preceq V^{\frac{k+1}{k}}$$

Proof. It suffices to show that there is a c such that if $\alpha\colon S^k \to G$ is a Lipschitz map with sufficiently large volume, then there is filling β of α such that:

$$\text{vol } \beta \leq c(\text{vol } \alpha)^{\frac{k+1}{k}}$$

Then by the lemmas 2.6-2.9 we know that there are $(\tau, s^i_2 \circ f)$-admissible maps $P_i(\alpha)\colon S^k \to X$ and horizontal homotopies $R_i(\alpha)$ connecting them. We will construct a disc filling by connecting the $R_i(\alpha)$ for $i = 0, \ldots, i_0$; this gives a homotopy between $P_0(\alpha) = P_\tau(\alpha)$ and $P_{i_0}(\alpha)$. The map $P_0(\alpha)$ is homotopic to α by the homotopy $Q_\tau(\alpha)$. Since $P_{i_0}(\alpha)$ is $(\tau, s_{2^i} \circ f)$-admissible, the map $\alpha' = f^{-1} \circ s_{2^{-i}} \circ P_{i_0}(\alpha)$ is admissible, but if i_0 is sufficiently large, we have that vol $\alpha' <$ vol Δ^k this implies that the image of α' lies entirely in the $(k-1)$-skeleton of τ, so vol $P_{i_0}(\alpha) = 0$. There is thus a homotopy contracting $P_{i_0}(\alpha) = 0$. There is thus a homotopy contracting $P_{i_0}(\alpha)$. Combining all these homotopies we get a homotopy from α to a point whose volume is $\preceq (\text{vol } \alpha)^{\frac{k+1}{k}}$. This homotopy is the required disc filling of α. □

2.4 More Upper Bounds

When G has no horizontal $(k+1)$-manifolds, these constructions still provide fillings of cycles, but these fillings satisfy weaker bounds. In some cases, however(especially when there are many horizontal k-manifolds), these weaker bounds may still be sharp.

Lemma 2.11. Let $k > 0$. Let $\tau, \eta, \phi\colon \tau \to G$, and $\psi\colon \eta \to G$ as in lemma 2.4, except with no requirement that ϕ and ψ be horizontal. Let $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ a function such that for any $(k+1)$-simplex Δ of η, we have mass $s_t(\psi(\Delta)) \leq f(t)$. If, as in lemma 2.4, we define:

$$R_i(\alpha) = s_{2^i}(R_{\psi(\eta)}(s_{2^{-i}}(\alpha)))$$

there is a c such that for all $i \geq 0$ and for all k-cycles α, we have that:

$$\partial R_i(\alpha) \leq P_{i+1}(\alpha) - P_i(\alpha)$$

and

$$\text{mass } R_i(\alpha) \leq 2^{-ki} c f(2^i) \text{mass } \alpha$$

Proof. The fact that:

$$\partial R_i(\alpha) = P_{i+1}(\alpha) - P_i(\alpha)$$

follows from the proof of Lemma 2.4.

If β is a simplicial chain in a simplicial complex σ, it can be written as:

$$\beta = \sum_{\Delta \subset \sigma} b_\Delta \Delta$$

Define:

$$||\beta||_1 = \sum_{\Delta \subset \sigma} |b_\Delta|$$

Recall that if γ is a Lipschitz k-cycle, then we defined:

$$R_{\psi(\eta)}(\gamma) = \psi_\#(P_\eta(X(\gamma)))$$

By Theorem 2, there is a c depending only on η such that $||P_\eta(X(\gamma))||_1 \leq c\text{mass } \gamma$. Therefore, we have that:

$$||P_\eta(X(s_{2^{-i}}(\alpha)))||_1 \leq c 2^{-ki} \text{mass } \alpha$$

so $R_i(\alpha)$ is the sum of at most $(c2^{-ki}\text{mass } \alpha)$ singular simplices of the form $(s_{2^i} \circ \psi)_{\#}(\Delta)$, where Δ is a simplex of η. Each of these has mass $\leq f(2^i)$, so:

$$\text{mass } R_i(\alpha) \leq 2^{-ki} c f(2^i) \text{mass } \alpha$$

□

In a similar spirit as Theorem 2.5 we have that:

Theorem 2.12. (R.Young) Let k, τ, ϕ, η and ψ satisfy the hypotheses of Lemma 2.11. Then:

$$FV_G^{k+1}(V) \preceq \sum_{i=1}^{(\log_2 V)/k} f(2^i) \frac{V}{2^k i}$$

So if $f(t) \sim t^p$, where $p > k$, we have:

$$FV_G^{k+1}(V) \preceq V^{\frac{p}{k}}$$

Chapter 3

Applications

3.1 Constructing horizontal triangulations

Theorems 2.5 and 2.10 show that when certain triangulations and certain k-horizontal maps exists, then G satisfies euclidean filling inequalities. In this section we will give a family of groups, for which such maps exists.

Definition 3.1. Given G a 2-step nilpotent group with center Z, we define the **central product** $G \times_Z G$ to be the quotient $G \times G/\sim$ where $\sim$ is the relation which identifies the centers of the two copies of G(i.e., the relation $(z, 1) \sim (1, z)$ for all $z \in Z$). Likewise, we define the n-fold central product to be the quotient:

$$G^{\times_Z n} = G^n / \sim$$

where $\sim$ is the relation:

$$(z, 1, 1, \ldots) \sim (1, z, 1, \ldots) \sim (1, 1, z, \ldots) \sim \ldots \sim (1, 1, 1, \ldots, z).$$

For all z in Z.

In this section we will prove quadratic bounds for Dehn functions of a certain family of central products of free nilpotent groups. Consider the free 2-step nilpotent group on r generators, which is given by the presentation:

$$\Lambda_r = \langle g_1, \ldots, g_r | [g_i, [g_j, g_k]] = 0 \text{ for all } 1 \leq i, j, k \leq r \rangle$$

Its abelianization is $\mathbb{Z}^r$, generated by the $g_i' s$, and its center isomorphic to $\mathbb{Z}^{\binom{r}{2}}$ generate by the elements of the form $[g_i, g_j], 1 \leq i < j \leq r$. Define:

$$\Lambda_{r,n} = (\Lambda_r)^{\times_Z n}$$

We claim that:

Proposition 3.2. (R.Young) $\Lambda_{r,n}$ has a quadratic Dehn function when $n \geq 2$.

This proposition was first stated without proof by Ol'shanskii and Sapir[8]. It will follow from applying theorem 2.10 with $k = 1$. We will show that $\Lambda_{r,n}$ satisfies the conditions of the theorem by finding a presentation of $\Lambda_{r,n}$, and then using that presentation to construct triangulations and horizontal maps.

These groups are lattices in nilpotent Lie groups: Λ_r is a lattice in the free 2-step nilpotent Lie group of rank r, which we we call F_r, and $\Lambda_{r,n}$ is a lattice in $F_{r,n} = (F_r)^{\times z n}$. Let $\mathfrak{f}_r$ be the Lie algebra of F_r, and let $v_i = \log g_i \in \mathfrak{f}_r$. If we define generators of $\mathfrak{f}_r$ by $v_i = \log g_i \in \mathfrak{f}_r$, then $\mathfrak{f}_r$ has a grading:

$$\mathfrak{f}_r = V_1 \oplus V_2 = \langle v_1, \dots, v_r \rangle \oplus \langle [v_i, v_j] \text{ for all } 1 \leq i < j \leq r \rangle$$

If $g_i j$ is the generator of the j-th factor in the central product $F_{r,n}$ and $\mathfrak{f}_{r,n}$ in its Lie algebra, we can likewise define a grading:

$$\mathfrak{f}_{r,n} = V_1^n \oplus V_2 = \langle v_{11}, \dots, v_{rn} \rangle \oplus \langle [v_{i1}, v_{j1}] | \text{ for all } 1 \leq i < j \leq r \rangle$$

For ease of notation, we will start by considering $\Lambda_{r,2}$. Let $g_1, \dots, g_r$ in $\Lambda_r \times \Lambda_r$ be the generators of the firs factor and let $h_1, \dots, h_r$ in $\Lambda_r \times \Lambda_r$ be the generators of the second factor. Then $\Lambda_{r,2}$ is given by the presentation:

$$\begin{aligned} \Lambda_{r,2} = \langle g_1, \dots, g_r, h_1, \dots, h_r | [g_i, [g_j, g_k]] & \text{ for all } i, j, k, && (1) \\ [h_i, [h_j, h_k]] & \text{ for all } i, j, k, && (2) \\ [g_i, h_j] & \text{ for all } i, j, && (3) \\ [g_i, g_j][h_i, h_j] & \text{ for all } i, j \rangle. && (4) \end{aligned}$$

We claim then that:

Lemma 3.3. $\Lambda_{r,2}$ can be presented as:

$$\begin{aligned} \Lambda_{r,2} = \langle g_1, \dots, g_r, h_1, \dots, h_r | [g_i, h_j] & \text{ for all } 1 \leq i, j \leq r, && (5) \\ [g_i h_j, g_j h_i] & \text{ for all} 1 \leq i, j \leq r \rangle && (6) \end{aligned}$$

Proof. We need to show that the relation $[g_i h_j, g_j h_i] = id$ holds in $\Lambda_{r,2}$, for all i, j and we need to show that relations $(1), (2), (4)$ can be deduced from (5) and (6). First, we reduce $[g_i h_j, g_j h_i]$ to the empty word ε by using $(1) - (4)$:

$$\begin{aligned} [g_i h_j, g_j h_i] &\to [g_i, g_j][h_j, h_i] \text{ by } (3) \\ &\to \varepsilon \text{ by } (4). \end{aligned}$$

In the first step, we shuffled all the g_i's and g_j's in $[g_ih_j, g_jh_i]$ to the beginning using the fact that g's and h's commute. Now we deduce $(1), (2), (4)$ from $(5), (6)$. First, note that (4) follows from $(5), (6)$ from the reverse of the argument above:

$$[g_i, g_j][h_j, h_i] \to [g_ih_j, g_jh_i] \to \varepsilon$$

Next, we cand reduce (1) as follows:

$$\begin{aligned}[g_i, [g_j, g_k]] &\to [g_i, [h_j, h_k]] \text{ by } (4)\\ &\to \varepsilon\end{aligned}$$

Here we reduced $[g_i, [h_j, h_k]]$ to the trivial word by using the fact that g_i commutes with each letter of $[h_j, h_k]$. The same procedure can be used to deduce (2). □

In a similar spirit, $\Lambda_{r,n}$ has also a similar presentation:

Lemma 3.4. $\Lambda_{r,n}$ can be presented as:

$$\begin{aligned}\Lambda_{r,n} = \langle g_{ij}, i = 1, \ldots, r, j = 1 \ldots, n | [g_{ij}, g_{kl}] &\text{ for all } j \neq l\\ [g_{ij}g_{kl}, g_{kj}g_{il}] &\text{ for all } j \neq l\rangle.\end{aligned}$$

The proof being similar to the one of lemma 3.3

Now in order to consctruct the triangulations and horizontal maps required we will use this presentation. Recall then that if Γ is a finitely presented group with presentation:

$$\Gamma = \langle x_1, \ldots, x_d | r_1, \ldots, r_s\rangle$$

then its Cayley complex X_Γ is a simply-connected 2-complex on which Γ acts geometrically(that is, properly discontinuously, cocompactly, and by isometries). The 1-skeleton of X_Γ is given by the Cayley graph of Γ with respect to the $\{x_i\}$. At each vertex of the Cayley graph of Γ, there is a loop corresponding to each relator r_i, and we obtain X_Γ by gluing a 2-cell to each such loop. Since we started with a presentation of Γ, this procedure results in a simply-connected complex. The advantage of the presentation in Lemma 3.4 is that there is a horizontal map from its Cayley complex

$X = X_{\Gamma_{r,n}}$ to $F_{r,n}$, which we will denote $h\colon X \to F_{r,n}$. Vertices of X correspond to elements of $\Gamma_{r,n}$ so we map each vertex to the corresponding element of $F_{r,n}$. Each edge e of X connects g and $gg_{ij}^{\pm 1}$ for some $g \in \Lambda_{r,n}$ and some i, j. Since $g_{ij} = \exp v_{ij}$ for all i, j, the points g and $gg_{ij}^{\pm 1}$ can be connected by a horizontal segment of the form $t \mapsto g \exp \pm t v_{ij}, 0 \leq t \leq 1$; we define $h(e)$ to be this segment. Then if $w = w_1 \dots w_p$ is a word, it corresponds to an edge path in X and its image under h is a horizontal curve γ_w in G which connects the points $e, w_1, w_1 w_2, \dots, w_1 w_2 \dots w_p$. Each relation in Lemma 3.4 then corresponds to a horizontal closed curve γ_w. We complete the definition of h by filling these curves with horizontal discs.

First, we consider $w = [g_{ij}, g_{kl}]$. The curve γ_w lies in the 2-parameter subgroup $\exp\langle v_{ij}, v_{kl}\rangle$, and we can fill it with a disc of the form $\exp(s v_{ij} + t v_{kl}),\ 0 \leq s, t \leq 1$. Next we consider $w = [g_{ij} g_{kl}, g_{kj} g_{il}]$. Here, the disc is a little more complicated, as is shown in Figure 3.1:

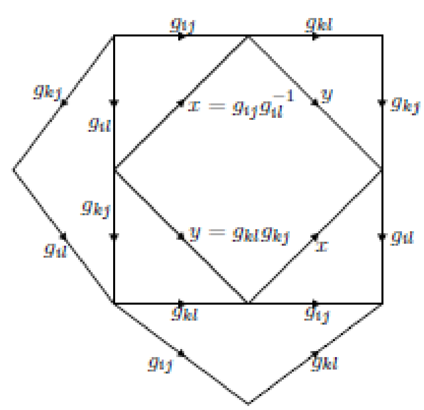


Figure 3.1: A disc filling $\gamma_w = [g_{ij} g_{kl}, g_{kj} g_{il}]$

The disc is made up of three quadrilaterals and four triangles. Each edge in the disc is a segment of a translate of a horizontal 1-parameter subgroup, and each face lies in a translate of a 2-dimensional horizontal subgroup of $F_{r,n}$. The quadrilateral on the left lies in a translate of the subgroup with Lie algebra $\langle v_{il}, v_{kj}\rangle$, the one on the bottom lies in a translate of $\exp\langle v_{kl}, v_{ij}\rangle$ and the one in the center lies in a translate of $\exp\langle v_{ij} - v_{il}, v_{kl} + v_{kj}\rangle$. Similarly, each triangle lies in a translate of either $\exp\langle v_{ij}, v_{il}\rangle$ or $\exp\langle v_{kj}, v_{kl}\rangle$. All of these subgroups are horizontal submanifolds of $F_{r,n}$ so the disc is horizontal.

As a consequence we conclude the following:

Lemma 3.5. If w is a word in $\Lambda_{r,n}$ which represents the identity, then there is a horizontal disc filling γ_w in $F_{r,n}$.

Proof. Since w represents the identity, it corresponds to a closed edge path in X, and since X is simply connected, this path can be filled by a disc in X. The image of this disc in $F_{r,n}$ is a disc filling γ_w. □

We can use this lemma to construct triangulations and maps satisfying the hypotheses of Theorem 2.10 and Theorem 2.5 with $k = 1$. To apply the theorem to a group G, we need a triangulation η of $G \times [0, 1]$ which restricts to triangulations τ_0 and τ_1 on $G \times \{0\}$ and $G \times \{1\}$ respectively, where τ_1 is a scaling of τ_0. In the following lemma, we show that if G contains a lattice Γ such that $s_2(\Gamma) \subset \Gamma$, then τ_0 can be chosen to be Γ-equivariant and η can be chosen to be a $s_2(\Gamma)$-equivariant.

Lemma 3.6. Given G a Carnot group, Γ a lattice in G such that $s_2(\Gamma) \subset \Gamma$ and let $(\tau, f\colon \tau \to G)$ be a Γ-equivariant triangulation. Let $(\tau_i, f_i\colon \tau_i \to G)$, $i = 0, 1$ be triangulations such that $\tau_0 = \tau_1 = \tau$, $f_0 = f$, and $f_1 = s_2 \circ f_0$. Then there is an $s_2(\Gamma)$-equivariant triangulation $(\eta, g\colon \eta \to G \times [0, 1])$ of $G \times [0, 1]$ which restricts to τ_i on $G \times \{i\}$ for $i = 0, 1$.

Proof. First, we construct $s_2(\Gamma)$-actions on τ_0 and τ_1 which make them equivariant triangulations. Let $\rho(g)\colon \tau \to \tau$, $g \in \Gamma$ be the action of Γ on τ. Define $\rho_0(g) = \rho(g)$ and $\rho_1(g) = \rho(s_{1/2}(g))$ for all g in $s_2(\Gamma)$. Then f_0 and f_1 are equivariant with respect to ρ_0 and ρ_1 respectively. In particular τ_0 and τ_1 descend to triangulations of $M = s_2(\Gamma)\diagdown G$. Since this is a smooth manifold, we can construct a triangulation $(\overline{\eta}, \overline{g}\colon \overline{\eta} \to M \times [0, 1])$ which restricts to $s_2(\Gamma)\diagdown\tau_i$ on $M \times \{i\}$. This lifts to the required triangulation of $G \times [0, 1]$. □

Since $g_{ij} = \exp v_{ij}$ and v_{ij} is in the first layer of the grading of $\mathfrak{f}_{r,n}$, we have $s_2(g_{ij}) = \exp(2v_{ij}) = g_{ij}^2$ and thus $s_2(\Lambda_{r,n}) \subset \Lambda_{r,n}$. The lemma thus applies to $\Lambda_{r,n}$, and we can get τ and η triangulations of $F_{r,n}$ and $F_{r,n} \times [0, 1]$ given by the Lemma 3.6.

We now define ϕ. For each vertex v of τ, let $\phi(v)$ be an element of $\Lambda_{r,n}$; we can choose these elements in an $\Gamma_{r,n}$-equivariant way. If e is an edge of τ with vertices x and y, we can choose a word $w = w(\phi(x)^{-1}\phi(y))$ in $\Gamma_{r,n}$ which represents $\phi(x)^{-1}\phi(y)$. Then γ_w is a curve connecting e to

$\phi(x)^{-1}\phi(y)$, and we can define $\phi(e)$ as the translation $\phi(x)\gamma_w$. Since w depends only on $\phi(x)^{-1}\phi(y)$ this definition is also $\Gamma_{r,n}$-equivariant. The map ϕ send the boundary of each 2-cell Δ of τ to a curve $g \cdot \gamma_{w'(\Delta)}$, where $w'(ds)$ is a word representing the identity. By lemma 3.5, we can extend ϕ to a horizontal map on Δ, and moreover, we can do this in an equivariant way. Doing this for every cell in the 2-skeleton of τ gives us an equivariant horizontal map on $\tau^(2)$, and we can extend it to an equivariant 2-horizontal map on all of τ.

Next we define ψ in a similar fashion. This time, because of the relationship between τ and η, many of our choices are already made for us. Let τ_i and f_i, $i = 0, 1$ be as in Lemma 3.6 and let $\iota_i \colon \tau_i \to \eta, i = 0, 1$, be the inclusions of the τ_i into the η. Let:

$$\psi_{\iota_0(\tau_0)} = \phi \circ \iota_0^{-1}$$

and

$$\psi_{\iota_1(\tau_1)} = s_2 \circ \iota_1^{-1}$$

These definitions are $s_2(\Lambda_{r,n})$-equivariant and if v is a vertex of $\iota_i(\tau_i), i = 0, 1$, then $\phi(v) \in \Lambda_{r,n}$. We extend ψ to the rest of the 1-skeleton of η in the same way as before. For each vertex v of η which is not in τ_0 or τ_1, we choose an element $\psi(v) \in \Lambda_{r,n}$. For each edge $e = (x, y)$ of η which is not in τ_0 or τ_1, we let:

$$\psi(e) = \psi(x) \cdot \gamma_{w(\psi^{-1}(x)\psi(y))}$$

Then if e is an edge of η, then $\psi(e)$ is a reparametrization of γ_w for some word w; this is true by construction if e is not in τ_1, and if e is an edge of τ_1, then $\psi(e)$ is a curve of the form $s_2 \circ \gamma_w$ for some $w = w_1 \dots w_p$. Since each of the curves making up γ_w is a segment of a horizontal one-parameter subgroup, $s_2 \circ \gamma_w$ is a reparametrization of γ_z, where $z = w_1^2 \dots w_p^2$. Thus ψ sends the boundary of each 2-cell Δ of η to a reparametrization of a curve $g \cdot \gamma_{w'(\Delta)}$ where w' is a word representing the identity. Using Lemma 3.5 we can extend ψ to a 2-horizontal map on all of η. All this can be done in an equivariant and Lipschitz way.

In conclusion, we have found triangulations and adequate horizontal maps that together with the Theorem 2.5 gives us a quadratic bound on the filling volume function and Dehn function of $\Lambda_{r,n}$

3.2 Finding other discs

We notice that using Theorem 2.12 and following the ideas of the previous chapter we can also get bounds on nilpotent groups of higher steps. The only part that remains to be seen is an analogous version of Lemma 3.5. For that we need to find adequate discs that fill the relations of our given group. For that reason we will see first how to fill other discs in higher step nilpotent Carnot groups(not only 2-nilpotent).

Now, given G a simply connected Carnot Lie group, and $\mathfrak{g}$ the respective Lie algebra with a Carnot grading. Let X, Y be any two elements of $\mathfrak{g}$ and x, y their respectives exponentials. The map we will consider is $H\colon [0,1] \times [0,1] \to G$ given by:

$$H(s,t) = \exp(sX)\exp(tY)$$

Now, as we have seen in the first parts, we need to consider the differential of such a map in order to obtain good bounds on the Dehn function of the group.

$$\begin{aligned}
DH_{(s,t)}(1,0) &= D(L_{H(s,t)})_e \left(\frac{d}{dh}\Big|_{h=0} \exp(-tY)\exp(-sX)\exp((s+h)X)\exp(tY) \right) \\
&= D(L_{H(s,t)})_e \left(\frac{d}{dh}\Big|_{h=0} \exp(-tY)\exp(hX)\exp(tY) \right) \\
&= D(L_{H(s,t)})_e \left(Ad(\exp(-tY))X \right) \\
&= D(L_{H(s,t)})_e \left(\exp(ad(-tY))X \right) \\
DH_{(s,t)}(0,1) &= D(L_{H(s,t)})_e \left(\frac{d}{dh}\Big|_{h=0} \exp(-tY)\exp(-sX)\exp(sX)\exp((t+h)Y) \right) \\
&= D(L_{H(s,t)})_e \left(\frac{d}{dh}\Big|_{h=0} \exp(hY) \right) \\
&= D(L_{H(s,t)})_e \left(Y \right)
\end{aligned}$$

Also note that:

$$\exp(ad(-tY))X = X - t[Y,X] + \frac{t^2}{2}[Y,[Y,X]] - \frac{t^3}{6}[Y,[Y,[Y,X]]] + \dots \quad (3.1)$$

So the image of the derivative of H lives in the left-invariant distribution spanned by Y and $\exp(ad(-tY))X$. In many cases this will give us a good bound on the distorsion of the dilation automorphism over these maps.

The fact that the exponential of the Lie group G is a diffeomorphism implies that we don't have too much freedom in choosing paths between points of the Lie group G. More especifically we have that:

Lemma 3.7. Let x, y be two elements in the Lie group G and define $A = \log(x^{-1}y)$ and $B = \log(yx^{-1})$. The two natural choices for paths between x and y; the paths $\gamma_1(t) = x\exp(tA)$ and $\gamma_2(t) = \exp(tB)x$ are the same.

Proof. It is clear that proving that those two paths are the same is equivalent to prove that $x\exp(tA)x^{-1} = \exp(tB)$. It is clear that these maps are 1-parameter subgroups of G that coincide at $t = 1$. As the exponential map is inyective this implies that the subgroups are the same, which implies the wanted result. □

3.3 3-Nilpotent groups

Let $\mathfrak{f}_n$ be the free 3-nilpotent Lie algebra of rank n. This Lie algebra is of the form:

$$\mathfrak{f}_n = \langle X_1, \dots, X_n\rangle \oplus \langle [X_i, X_j], 1 \le i < j \le n\rangle \oplus \langle [X_i, [X_j, X_k]], 1 \le i, j, k \le n\rangle$$

Note that $\mathfrak{z}(\mathfrak{f}_n) = \langle [X_i, [X_j, X_k]], 1 \le i, j, k \le n\rangle$. Now consider the Lie algebra $\mathfrak{g}_n = \dfrac{\mathfrak{f}_n \times \mathfrak{f}_n}{W}$ where $W = \{(z, z) | z \in \mathfrak{z}(\mathfrak{f}_n)\}$. This Lie algebra is also Carnot graded. Define now $\mathfrak{h}_n$ as the Lie sub-algebra of $\mathfrak{g}_n$ spanned as a Lie algebra by $A_i = X_i + Y_i$ for all i. Then we have that:

$$\mathfrak{h}_n = \langle X_1 + Y_1, \dots, X_n + Y_n\rangle \oplus \langle [X_i, X_j] + [Y_i, Y_j] | 1 \le i < j \le n\rangle$$

It is clear that $\mathfrak{h}_n$ is 2-nilpotent. Consider the simply connected nilpotent group G_n which has as Lie algebra $\mathfrak{g}_n$; and H_n the Lie subgroup of G_n which has as Lie algebra $\mathfrak{h}_n$.

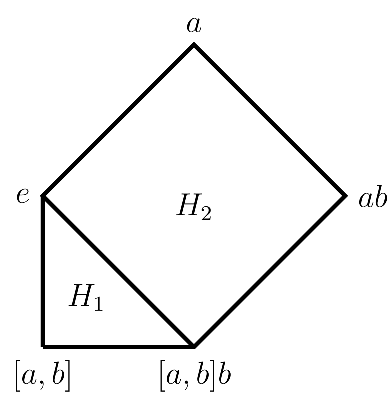
Let $\Gamma_n = \langle g_1, g_2, \dots, g_n, h_1, h_2, \dots, h_n\rangle$ the subgroup generated by $g_i = \exp X_i$, $h_j = \exp Y_j$ if we somehow manage to prove that this group is a lattice in G_n and is given by the following presentation:

$$\Gamma_n = \langle g_1, g_2, \dots, g_n, h_1, h_2, \dots, h_n | [g_i, h_j], [g_i h_i, [g_j h_j, g_k h_k]] \text{ for all } i, j, k\rangle$$

Then we get a cubic upper bound on the Dehn function of this group.

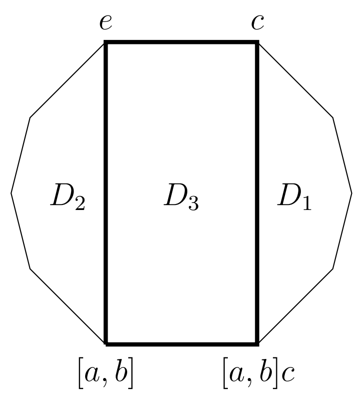
Theorem 3.8. Γ_n has a cubic Dehn function.

Proof. Following the method of Young, we need to fill by adequate discs the relations of the group Γ_n. As we have already seen the relations of the form $[g_i, h_j]$ are filled by horizontal discs, we only need to worry about the relations of the form $[g_i h_i, [g_j h_j, g_k h_k]]$. Without loss of generality we will call $g_i h_i = c$, $g_j h_j = a$, $g_k h_k = b$. We will divide the construction of the filling in two parts; first, we will fill the word $[a, b]$ and after that we will fill the word $[c, [a, b]]$.



where $H_1(s, t) = \exp(s[A, B]) \exp(tB)$ is only defined in the triangle $\{(s, t) \in [0, 1] \times [0, 1] : t \leq s\}$(here the fact that $[A, B]$ and B commute is really important) and $H_2(s, t) = \exp(sC) \exp(tA)$, C being $\log(aba^{-1})$ now this two disks have the property that their differential is in a space that is spanned by a horizontal vector and a vector in $\mathfrak{h}$ (this is because aba^{-1} is in H), so it lives in $V_1 \oplus V_2$.

Now in order to fill the word $[c, [a, b]]$ we will use the fact that c and $[a, b]$ commute; then we have the following diagram:



where the discs D_1 and D_2 are of the form seen in the last figure and D_3 is given by the same formula seen in the section 3.2. Now we have seen that we can cover any relationship by a disc that has derivatives in an adequate distribution.

This and the Theorem 2.12 implies that we have a cubic bound on the Dehn function and flling volume function of this group.

□

3.4 General Case

Let $\mathfrak{f}_{p,n}$ be the free p-nilpotent Lie algebra of rank n. This Lie algebra is of the form:

$$\mathfrak{f}_{p,n} = \langle X_1, \ldots, X_n\rangle \oplus \langle [X_i, X_j], 1 \leq i, j \leq n\rangle \oplus \ldots \oplus \langle [X_{i_1}, \ldots, X_{i_p}] 1 \leq i_1, \ldots, i_p \leq n\rangle$$

We will consider a central product of these Lie algebras and give an upper bound on the Dehn function of these central products. Let's consider $F_{p,n}$ the simply connected Lie group with Lie algebra $\mathfrak{f}_{p,n}$ and $G_{p,n} = F_{p,n} \times_{id} F_{p,n}$ the simply connected Lie group with Lie algebra $\mathfrak{g}_{p,n} = \mathfrak{f}_{p,n} \times \mathfrak{f}_{p,n}/\{(x, -x) | x \in \mathfrak{z}(\mathfrak{f}_{p,n})\}$.

It can be seen that $G_{p,n}$ has a lattice $\Lambda_{p,n}$ with a presentation:

$$\mathcal{P}(\Lambda_{p,n}) = \left\langle \begin{array}{cccc} x_1, & x_2, \ldots & x_n \\ y_1, & y_2, \ldots & y_n \end{array} \;\middle|\; \begin{array}{l} \left[x_{i_1}y_{i_1}, x_{i_2}y_{i_2}, \ldots, x_{i_{p-1}}y_{i_{p-1}}\right], 1 \leq i_1, \ldots, i_{p-1} \leq n \\ \left[x_i, y_j\right], 1 \leq i, j \leq n \end{array} \right\rangle$$

Now following R.Young ideas([Scaled Relators] and [Filling Inequalities]) we can obtain upper bounds on the Dehn function of $\Lambda_{p,n}$, by filling the paths given by the relations by a disc that behaves well under dilations of $G_{p,n}$.

3.5 Jet Groups

Other consequence of the theorem 2.5 is actually getting bounds for the higher Dehn functions of Heisenberg groups. This is done by proving the bound for the jet groups which we will define now. We will mainly consider the m-jet bundle of $\mathbb{R}^k$, which we denote by $J^m(\mathbb{R}^k)$. This is a vector bundle which is a generalization of the cotangent bundle and is often used to describe differential relations. A smooth map from $\mathbb{R}^k$ to $\mathbb{R}$ has a differential which can be considered as a map from $\mathbb{R}^k$ to the cotangent bundle $T(\mathbb{R}^k)^*$. Likewise, its m-th order derivative can be considered as a map to the i-th symmetric power of $T(\mathbb{R}^k)^*$. The m-jet bundle is a sum of these symmetric powers. Moreover, it is a trivial vector bundle over $\mathbb{R}^k$ with fiber:

$$W = \bigoplus_{i=0}^{m} W_i$$

where $W_i = \bigodot^i(\mathbb{R}^k)^*$ is the ith symmetric power of $(R^k)^*$ and $W_0 = \bigodot^i(\mathbb{R}^k)^* = \mathbb{R}$. An application $f\colon \mathbb{R}^k \to \mathbb{R}$ of class $\mathcal{C}^n$ corresponds to a section of class $\mathcal{C}^{n-m}$, called a **prolongation**, $j^m(f)\colon \mathbb{R}^k \to J^m(\mathbb{R}^k)$, given by taking derivatives: the projection to W_0 corresponds to the original application, the projection to $W_1 = (\mathbb{R}^k)^*$ is the gradient of f and so on. We will often write $j^m_p(f)$ in place of $j^m(f)(p)$. In the Carnot structure that we will construct on $J^m(\mathbb{R}^k)$ prolongations of smooth functions will be horizontal.

In [3],4.1.D,4.4.A-B], Gromov used infinitesimal invertibility and the h-principle to prove a Lipschitz extension theorem for many spaces including the k-jet bundle. This theorem can be used to construct maps and triangulations satisfying Theorem 2.5 for many groups. In the case of the k-jet bundle we can construct these maps and these triangulations fairly explicitly. Moreover this family of jet groups includes the higher dimensional Heisenberg groups and the filiform groups.

This family of groups has also appeared as a family of non-rigid Carnot groups[4]. Warhurst defined the group by putting a group structure on the m-jet bundle $J^m(\mathbb{R}^k)$, which we will describe describe now:

Using a basis of $\mathbb{R}^k$ we can construct a basis of W. Let $\{e_1, e_2, \ldots, e_k\}$ be the standard basis of $\mathbb{R}^k$, and let $\{e_1^*, e_2^*, \ldots, e_k^*\}$ be the corresponding dual basis of $(\mathbb{R}^k)^*$. If we let:

$$y_{(a_1,a_2,\ldots,a_k)} = (e_1^*)^{a_1} \odot (e_2^*)^{a_2} \odot \ldots \odot (e_k^*)^{a_k}.$$

then:

$$\{y_{(a_1,a_2,\ldots,a_k)} | \sum a_i = n\}$$

is a basis of W_n, and:

$$\{y_{(a_1,a_2,\ldots,a_k)} | \sum a_i \leq m\}$$

is a basis of W. Using this basis we can write $J^m(\mathbb{R}^k)$ as a product $J^m(\mathbb{R}^k) = \mathbb{R}^k \times W$. Now, note that for every $p \in J^m(\mathbb{R}^k)$, there is a unique polynomial in k variables of degree at most m whose prolongation passes through p. We will call this polynomial $P(p)$, and we use it to construct an action of $\mathbb{R}^k$ on W. Let's denote $D_x^m(f) \in W$ to be the first m derivatives of f at x, so that:

$$j_x^m(f) = (x, D_x^m(f))$$

If $x \in \mathbb{R}^k$ and $w \in W$, we let $S_x \colon W \to W$ be the map $S_x(w) = D_x^m(P((0,w)))$. This is an action of $\mathbb{R}^k$ on W, and we define:

$$(p_1, p_2).(q_1, q_2) = (p_1 + q_1, S_{q_1}(p_2) + q_2)$$

This makes $J^m(\mathbb{R}^k)$ a semidirect product of $\mathbb{R}^k$ and W. One can check that $\mathbb{R}^k \times \{0\}$ is a subgroup and that the translate $p.(\mathbb{R}^k \times \{0\})$ is the graph of $j^m(P(p))$.

To describe the Lie algebra $\mathfrak{j}_{m,k}$ of $J^m(\mathbb{R}^k)$, consider the basis

$$\{y_{(a_1,\ldots,a_k)} | \sum a_i \leq m\} \cup \{e_1, e_2, \ldots, e_k\}$$

of $\mathfrak{j}_{m,k}$. Calculating brackets, we find that:

$$[e_i, y_{(a_1,\ldots,a_k)}] = y_{(a_1,\ldots,a_i-1,\ldots,a_k)}$$

and all other brackets are zero. We can give $\mathfrak{j}_{m,k}$ the grading:

$$\mathfrak{j}_{m,k} = (\mathbb{R}^k \oplus W_m) \oplus W_{m-1} \ldots \oplus W_0$$

Moreover, since the structure constants of ${}_{m,k}$ with respect to the basis $\{x_i, y_{(a_1,\ldots,a_k)}\}$ are rational, $\{\exp x_1, \ldots, \exp x_k\} \cup \{y_{(a_1,\ldots,a_k)} | \sum a_i = m\}$ generate a lattice in $J^m(\mathbb{R}^k)$. We will call this lattice $\Gamma_{m,k}$. Since the generators are in $\exp(\mathbb{R}^k \oplus W_m)$, we have that $s_2(\Gamma_{m,k}) \subset \Gamma_{m,k}$.

Note that the groups $J^1(\mathbb{R}^k)$ are the $(2k+1)$-dimensional Heisnberg groups, and $J^2(\mathbb{R}^2)$ is the 3-step nilpotent group with quadratic Dehn function given in [9], one isomorphism between them takes:

$$a,\ b,\ c,\ d,\ e,\ f,\ g,\ h$$

to:

$$y_{(2,0)},\ x_1,\ y_{(1,1)},\ x_2,\ y_{(0,2)},\ -y_{(1,0)},\ y_{(0,1)},\ -y_{(0,0)}$$

respectively. Warhurst showed that the left-invariant plane field corresponding to $\mathbb{R}^k \oplus W_m$ agrees with the standard contact structure on $J^m(\mathbb{R}^k)$[4]. This gives a way to construct horizontal submanifolds: if U is an open subset of $\mathbb{R}^k$ and $f\colon U \to \mathbb{R}$ is smooth, we define M_f as the image of $j^m(f)\colon U \to J^m(\mathbb{R}^k)$. We will prove then that M_f is a horizontal submanifold:

Lemma 3.9. If U is an open subset of $\mathbb{R}^k$ and $f\colon U \to \mathbb{R}$ is smooth, then M_f is a smooth horizontal submanifold of $J^m(\mathbb{R}^k)$.

Proof. M_f is smooth by the definition of $j^m(f)$, so it just remains to show that its tangent plane lies in a translate of $V_1 = \mathbb{R}^k \oplus W_m$. First, note that any translate of a prolongation is still a prolongation. Specifically, if $p = (p_1, p_2) \in J^m(\mathbb{R}^k)$, then $p.M_f = M_g$ for $g\colon U' \to \mathbb{R}$ defined by:

$$g(x) = f(x - p_1) + P(p)(x)$$

where $U' = U + p_1 = \{u + p_1 |\ u \in U\}$ and $P(p)(x)$ is the polynomial used above to define the group structure on $J^m(\mathbb{R}^k)$. This is so, because for all $x \in U$ we have that:

$$\begin{aligned}(x + p_1, j^m g(x + p_1)) &= (x + p_1, j^m f(x) + j^m(P(p))(x + p_1)) \\ &= (x + p_1, D^m_{x+p_1}(P(p_1, p_2)) + j^m f(x)) \\ &= (x + p_1, D^m_x(P(0, p_2)) + j^m f(x)) \\ &= (x + p_1, S_x(p_2) + j^m f(x))\end{aligned}$$

and also that:

$$\begin{aligned}(p_1, p_2).(x, j^m f(x)) &= (x + p_1, S_x(p_2) + j^m f(x)) \\ &= (x + p_1, j^m g(x + p_1)).\end{aligned}$$

Now given $x \in U$ if we take $p = [j^m_x(f)]^{-1}$ in the previous construction, then g is a smooth function which vanishes to m-th order at 0. The tangent plane to M_f at $j^m_x(f)$ is then a translate of the tangent plane to M_g at 0. Since the first m derivatives of g disappear at 0, the tangent plane to M_g at 0 is contained in $\mathbb{R}^k \oplus W_m$. Indeed, if we consider the $D^m_0(g)$ as a map from $\mathbb{R}^k \to W_m$, then the tangent plane to M_g is the graph of this map. □

Now we define a class of horizontal manifolds coming from the construction of the lemma 3.9.

Definition 3.10. If U is an open subset of $\mathbb{R}^k$ and $f\colon U \to \mathbb{R}$ is a smooth map, we say that any submanifold Y of M_f is **holonomic**. If X is a complex, a Lipschitz map $f\colon X \to J^m(\mathbb{R}^k)$ is holonomic if and only if its image lies in M_f for some smooth $f\colon U \to \mathbb{R}$.

We then know by lemma 3.9 that holonomic submanifolds are horizontal. A cycle in a holonomic submanifold is equipped with a smooth function, and we can use this to construct a holonomic filling. That is, if $f\colon U \to \mathbb{R}^k$ is a smooth function defined on an open set and if α is a singular Lipschitz cycle in M_f, then we can construct a holonomic filling of α. The support of α is compact, so there is a smooth function $\overline{f}\colon \mathbb{R}^k \to \mathbb{R}$ which agrees with f on a neighbourhood of the support of α . In particular, we can consider α as $J^m(\overline{f})_\#(\alpha_0)$ for some cycle α_0 in $\mathbb{R}^k$. If β_0 is a chain in $\mathbb{R}^k$ which fills α_0, then $J^m(\overline{f})_\#(\beta_0)$ is a chain in $J^m(\mathbb{R}^k)$ which fills α.

In general, a horizontal map is not even necessarily even locally holonomic, and it can be difficult to fill an arbitrary horizontal map with an horizontal filling. For our purposes, it suffices to fill locally holonomic maps. To work with such maps, we will define *augmented maps* which are horizontal maps locally equipped with smooth functions.

Definition 3.11. Let X be a simplicial complex. Let $p_{\mathbb{R}^k}\colon J^m(\mathbb{R}^k) \to \mathbb{R}^k$ be the bundle projection. An augmented map from X to $J^m(\mathbb{R}^k)$ is a tuple:

$$\big(\alpha\colon X \to J^m(\mathbb{R}^k), \{f_\Delta\colon V_\Delta \to \mathbb{R}\}_{\Delta \subset X}\big)$$

which satisfies two conditions. First, we require that the map α is holonomic on each cell. That is, the image of a cell Δ is contained in M_{f_Δ}. Second, we require that if Δ_1 is a face of Δ_2, then $M_{f_{\Delta_1}} \subset M_{f_{\Delta_2}}$. Note that this imposes compatibility conditions on any pair of faces that intersect, because if Δ and Δ' intersect, then $M_{f_{\Delta\cap\Delta'}} \subset M_{f_\Delta}$ and $M_{f_{\Delta\cap\Delta'}} \subset M_{f_{\Delta'}}$, so f_Δ and f'_Δ agree on a neighbourhood of $\Delta \cap \Delta'$. If X is a subcomplex of Y and if $(\alpha, \{f_\Delta\}_{\Delta\subset X})$ is an augmented map on X, we say that $(\beta, \{g_\Delta\}_{\Delta\subset Y})$ *extends* $(\alpha, \{f_\Delta\}_{\Delta\subset X})$ if and only if β extends α in the ordinary sense and $M_{g_\Delta} \subset M_{f_\Delta}$ for all simplices $\Delta \subset X$. If $\kappa\colon X \to \mathbb{R}^k$ then we say that $(\alpha, \{f_\Delta\})$ covers κ if $p_{\mathbb{R}^k} \circ \alpha = \kappa$.

Lemma 3.12. If $\kappa\colon \Delta \to \mathbb{R}^k$ is a Lipschitz embedding of a simplex and

$$(\alpha, \{f_\delta\colon V_\delta \to \mathbb{R}\})$$

is an augmented map on $\partial\Delta$ which covers $\kappa\big|_{\partial\Delta}$, then there is an augmented map $(\beta, \{g_\delta\colon W_\delta \to \mathbb{R}\}_{\delta\subset\Delta})$ which extends $(\alpha, \{f_\delta\})$ and covers κ.

Proof. It suffices to find a smooth function $g\colon \mathbb{R}^k \to \mathbb{R}$ such that g agrees with $f\delta$ on a neighbourhood of $\kappa(\delta)$ for each face δ. If we have such a g, we can construct the required extension by letting $g_\delta = g$, $\beta = J^m(g) \circ \kappa$, and $W_\Delta = \mathbb{R}^k$. To find g, we use a partition of unity. If δ is a face of Δ, we define new domains:

$$U_\delta = V_\delta \setminus \bigcup_{\delta' \not\supset \delta} \kappa(\delta')$$

for all $\delta \subsetneq \Delta$. And for $\delta = \Delta$, we define:

$$U_\Delta = \mathbb{R}^k \setminus \kappa(\partial\Delta)$$

Now, for each δ, we have defined the U_δ in order that $\kappa(\text{int }\delta) \subset U_\delta$, so the U_δ's form an open cover of $\mathbb{R}^k$. Therefore there exists $\{\rho_\delta\colon \mathbb{R}^k \to \mathbb{R}\}_{\delta\subset\Delta}$ a smooth partition of unity subordinate to the cover $\{U_\delta\}$.

Let $f_\Delta\colon \mathbb{R}^k \to \mathbb{R}$ be a smooth function and let $g\colon \mathbb{R}^k \to \mathbb{R}$ be:

$$g = \sum_\delta \rho_\delta f_\delta$$

We claim that if δ is a simplex of Δ, then g agrees with f_δ on some neighbourhood of $\kappa(\delta)$. Now for a given x in $\kappa(\delta)$, we have that if $x \in \text{supp } \rho_{\delta'}$, then x is in $U_{\delta'}$ which implies that $U_{\delta'} \cap \kappa(\delta) \neq \emptyset$ when $\delta \not\supset \delta'$, so if $x \in \operatorname{supp} \rho_{\delta'}$, then $\delta' \subset \delta$. There is thus a neighbourhood U of x such that:

$$g\big|_U = \sum_{\delta' \subset \delta} \rho_{\delta'} f_{\delta'}$$

Furthermore, we know that all the $f_{\delta'}$ agree with $\mathfrak{f}_\delta$ on a neighbourhood of x. Since this is true for any $x \in \kappa(\delta)$, we conclude that f_δ and g agree on some neighbourhood of $\kappa(\delta)$. We can thus construct the required extension by letting $g_\Delta = g$ and $\beta = J^m(g) \circ \kappa$. □

This allows us to construct maps satisfying the conditions of Theorem 2.5 and Theorem 2.10 by induction on dimension. We first note that there is a natural action of $J^m(\mathbb{R}^k)$ on augmented maps. Given $p = (p_1, p_2)$ in $J^m(\mathbb{R}^k)$ we define:

$$p \cdot (\alpha, \{f_\Delta\}_{\Delta \subset X}) = (p \cdot \alpha, \{p \cdot f_\Delta\}_{\Delta \subset X})$$

where

$$\begin{aligned} (p \cdot \alpha)(x) &= p \cdot \alpha(x) \\ (p \cdot f_\Delta)(x) &= f_\Delta(x - p_1) + P(p)(x) \end{aligned}$$

It is easily seen that this is a group action on the space of augmented maps.

Lemma 3.13. There are τ, η, ϕ and ψ satisfying the conditions of Theorem 2.5 and Theorem 2.10 for $G = J^m(\mathbb{R}^{k+1})$ and $k = k$.

Proof. Let $\Gamma = \Gamma_{m,k+1}$ and recall that $s_2(\Gamma) \subset \Gamma$. Let $p\colon J^m(\mathbb{R}^{k+1}) \to \mathbb{R}^{k+1}$ be the bundle projection and let $(\tau, f\colon \tau \to G)$ be a Γ-equivariant triangulation of G and let $(\eta, g\colon \eta \to G \times [0,1])$ be a $s_2(\Gamma)$-equivariant triangulation of $G \times [0,1]$ as in Lemma 3.6. We will construct horizontal maps $\phi\colon \tau \to G$ and $\psi\colon \eta \to G$ by constructing augmented maps on the $(k+1)$-skeletons of τ and η. We can construct of Γ on $\mathbb{R}^{k+1}$ by letting the action of γ send $x \mapsto x + p(\gamma)$. After possibly subdividing τ, we can construct a Γ-equivariant map $\kappa\colon \tau \to \mathbb{R}^{k+1}$ so that the vertices of any simplex of τ lie in general position and each simplex is mapped linearly to $\mathbb{R}^{k+1}$; this is an embedding on each simplex in $\tau^{(k+1)}$. We can now use Lemma 3.12 to build a Γ-equivariant augmented map on the 0-skeleton, then the 1-skeleton, and on up to the $(k+1)$-skeleton. This constructs a horizontal

Γ-equivariant map on $\tau^{(k+1)}$ which can then be extended to all of τ. We construct ψ by a similar process. The main difference is the starting point; if τ_0 and τ_1 are as in Lemma 3.6 and if $\iota_i\colon \tau_i \to \eta,\ i = 0, 1$, are the inclusions of the τ_i into η, we define $\psi_0\colon \tau_0 \cup \tau_1 \to G$ by

$$\psi_0\big|_{\iota_0(\tau_0)} = \psi \circ \iota_0^{-1}$$

and

$$\psi_1\big|_{\iota_1(\tau_1)} = s_2 \circ \psi \circ \iota_1^{-1}$$

and extend this to an $s_2(\Gamma)$-equivariant $(k+1)$-horizontal map on η. $\square$

We then have

Theorem 3.14. (R.Young) $J^m(\mathbb{R}^k)$ satisfies the filling inequalities:

$$FV^n(V) \prec V^{\frac{n}{n-1}} \text{ for } 2 \leq n \leq k$$
$$\delta^{n-1}(V) \prec V^{\frac{n}{n-1}} \text{ for } 2 \leq n \leq k$$

In particular for the Heisenberg group $H_{2k+1} = J^1(\mathbb{R}^k)$, we have: $R.Young)\mathrm{H}_{2k+1}$ satisfies the filling inequalities:

$$FV^n(V) \prec V^{\frac{n}{n-1}} \text{ for } 2 \leq n \leq k$$
$$\delta^{n-1}(V) \prec V^{\frac{n}{n-1}} \text{ for } 2 \leq n \leq k$$

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